ON THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS WHOSE COEFFICIENTS DEPEND ONLY ON THE TIME VARIABLE

Seiichiro Wakabayashi

Abstract. In this paper we investigate the Cauchy problem for hyperbolic operators with triple characteristics whose coefficients depend only on the time variable. And we give sufficient conditions for C^∞ well-posedness. We shall also consider necessary conditions.

1. Introduction

In [12] we studies the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time-dependent coefficients. And we gave sufficient conditions for the Cauchy problem to be C^∞ well-posed under the assumption that the coefficients, for instance, are real analytic. These sufficient conditions are also necessary if the space dimension is less than 3, or if the coefficients are semi-algebraic functions with respect to the time variable. In [11] we considered the Cauchy problem for hyperbolic operators of third order with time-dependent coefficients and defined the sub-sub-principal symbols. We showed that the Cauchy problem is C^∞

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well-posed under some conditions on the subprincipal symbols and the sub-sub-principal symbols. In this paper we shall deal with hyperbolic operators with time-dependent coefficients and triple characteristics and give suficient conditions for the Cauchy problem to be C^∞ well-posed. Our results are extensions of the results given in [11] to higher-order hyperbolic operators. In doing so, we shall introduce new quantities as generalizations of "sub-sub-principal symbols." It will be proved that our sufficient conditions are also necessary for C^∞ well-posedness under additional conditions.

Let $m \in \mathbf{N}$ and $P(t,\tau,\xi) \equiv \tau^m + \sum_{j=1}^m \sum_{|\alpha|\le j} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1,\cdots,\xi_n)$ of degree m whose coefficients $a_{j,\alpha}(t)$ belong to $C^\infty([0,\infty])$. Here $\alpha = (\alpha_1,\cdots,\alpha_n) \in (\mathbf{Z}_+)^n$ is a multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, where $\mathbf{Z}_+ = \mathbf{N}\cup\{0\}\ (=\{0,1,2,3,\cdots\})$. We consider the Cauchy problem

$$\text{(CP)}\qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty)\times\mathbf{R}^n,\\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n\ (\ 0\le j\le m-1) \end{cases}$$

in the framework of the space of C^∞ functions, where $D_t = -i\partial/\partial t\ (=-i\partial_t)$, $D_x = (D_1,\cdots,D_n) = -i(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $f(t,x) \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j(x) \in C^\infty(\mathbf{R}^n)\ (\ 0\le j\le m-1)$.

Definition **1.1.** (i) The Cauchy problem (CP) is said to be C^∞ well-posed if the following conditions (E) and (U) are satisfied:

(E) For any $f \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)\ (\ 0\le j\le m-1)$ there is $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfying (CP).

(U) If $s>0$, $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$, $D_t^j u(t,x)|_{t=0} = 0\ (\ 0\le j\le m-1)$ and $P(t,D_t,D_x)u(t,x)$ vanishes for $t<s$, then $u(t,x)$ also vanishes for $t<s$.

(ii) We say that the Cauchy problem (CP) has finite propagation property (has finite propagation speeds) if the following condition (F) is satisfied:

(F) For any $T>0$ there is a convex closed cone Γ_T in $\mathbf{R}^n$ (with its vertex at the origin) such that $\Gamma_T \subset \{t>0\}\cup\{0\}$, and for any $(t_0,x^0)\in\mathbf{R}^{n+1}$ with $0<t_0\le T$

$$u = 0 \quad \text{in } \Gamma_T(t_0,x^0) \equiv \{(t_0,x^0)\} - \Gamma_T$$

if $u \in C^\infty(\mathbf{R}^{n+1})$, $\operatorname{supp} u \subset [0,\infty)\times\mathbf{R}^n$ and

$$P(t,D_t,D_x)u = 0 \quad \text{in } \Gamma_T(t_0,x^0).$$

We assume that the following conditions are satisfied:

(A-1) $a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j$, $j-1$) are real analytic on $[0,\infty)$.

(A-2) For some $\kappa_0 \in [1, 3/2)$ $a_{j,\alpha}(t) \in \mathcal{E}^{\{\kappa_0\}}([0,\infty))$ ($2 \le j \le m$, $|\alpha| = j-2$).

Here we say that $a(t) \in \mathcal{E}^{\{\kappa\}}(I)$ if for any $T > 0$ there are $h > 0$ and $C_T > 0$ satisfying

$$|\partial_t^k a(t)| \le C_T h^k (k!)^\kappa \quad \text{for } k \in \mathbf{Z}_+ \text{ and } t \in I \text{ with } |t| \le T,$$

where $1 \le \kappa < \infty$ and I is a closed interval of $\mathbf{R}$. From (A-1) there are a complex neighborhood Ω of $[0,\infty)$ (in $\mathbf{C}$) and $\delta_0 > 0$ such that $[-\delta_0, \infty) \subset \Omega$, $\Omega \cap \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \le T\}$ is compact for any $T > 0$, and $a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j$) are regarded as analytic functions defined in Ω. Put

$$p(t,\tau,\xi) = \tau^m + \sum_{j=1}^{m} \sum_{|\alpha|=j} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha \, (\equiv P_m(t,\tau,\xi)),$$

$$P_k(t,\tau,\xi) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha \quad (0 \le k \le m-1).$$

We also assume that the following conditions (H) and (T) are satisfied:

(H) $p(t,\tau,\xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \cdots, 0) \in \mathbf{R}^{n+1}$ for $t \in [-\delta_0, \infty)$, *i.e.*,

$$p(t, \tau - i, \xi) \ne 0 \quad \text{for any } (t,\tau,\xi) \in [-\delta_0, \infty) \times \mathbf{R} \times \mathbf{R}^n.$$

(T) The characteristic roots are at most triple, *i.e.*,

$$\partial_\tau^3 p(t,\tau,\xi) \ne 0 \quad \text{if } (t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times S^{n-1} \text{ and}$$
$$p(t,\tau,\xi) = \partial_\tau p(t,\tau,\xi) = \partial_\tau^2 p(t,\tau,\xi) = 0,$$

where $S^{n-1} = \{\xi \in \mathbf{R}^n;\ |\xi| = 1\}$. Let $\Gamma(p(t,\cdot,\cdot),\vartheta)$ be the connected component of the set $\{(\tau,\xi) \in \mathbf{R}^{n+1} \setminus \{0\};\ p(t,\tau,\xi) \ne 0\}$ which contains ϑ, and define the generalized flows $K^\pm_{(t_0,x^0)}$ for $p(t,\tau,\xi)$ by

$$\begin{aligned} K^\pm_{(t_0,x^0)} = \{ &(t(s), x(s)) \in [0,\infty) \times \mathbf{R}^n;\ \pm s \ge 0 \text{ and } \{(t(s),x(s))\} \text{ is} \\ &\text{a Lipschitz continuous curve in } [0,\infty) \times \mathbf{R}^n \text{ satisfying} \\ &(d/ds)(t(s),x(s)) \in \Gamma(p(t(s),\cdot,\cdot),\vartheta)^* \ (\ a.e.\ s) \text{ and} \end{aligned}$$

$$(t(0), x(0)) = (t_0, x^0)\},$$

where $(t_0, x^0) \in [0, \infty) \times \mathbf{R}^n$ and $\Gamma^* = \{(t, x) \in \mathbf{R}^{n+1};\ t\tau + x \cdot \xi \geq 0$ for any $(\tau, \xi) \in \Gamma\}$. $K^+_{(t_0,x^0)}$ (resp. $K^-_{(t_0,x^0)}$) gives an estimate of the influence domain (resp. the dependence domain) of (t_0, x^0) (see Theorem 1.2 below). To describe conditions on the lower order terms we define the polynomials $h_j(t, \tau, \xi)$ ($\equiv h_j(t, \tau, \xi; p)$) of (τ, ξ) by

$$|p(t, \tau - i\gamma, \xi)|^2 = \sum_{j=0}^{m} \gamma^{2j} h_{m-j}(t, \tau, \xi)$$
$$\text{for } (t, \tau, \xi) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}^n \text{ and } \gamma \in \mathbf{R}.$$

Since $|p(t, \tau - i\gamma, \xi)|^2 = \prod_{j=1}^m ((\tau - \lambda_j(t, \xi))^2 + \gamma^2)$, we have

$$(1.1) \qquad h_k(t, \tau, \xi) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq m} \prod_{l=1}^{k} (\tau - \lambda_{j_l}(t, \xi))^2 \quad (1 \leq k \leq m),$$

where $p(t, \tau, \xi) = \prod_{j=1}^m (\tau - \lambda_j(t, \xi))$. Let $\mathcal{R}(\xi)$ be a set-valued function, whose values are discrete subsets of $\mathbf{C}$, defined for $\xi \in S^{n-1}$ satisfying the following:

$$\begin{cases} \text{For any } T > 0 \text{ there is } N_T \in \mathbf{Z}_+ \text{ such that} \\ \#\{\lambda \in \mathcal{R}(\xi);\ \mathrm{Re}\,\lambda \in [0, T]\} \leq N_T \quad \text{for } \xi \in S^{n-1}. \end{cases}$$

Here $\#A$ denotes the number of the elements of a set A. We assume that $0 \in \mathcal{R}(\xi)$ when $\mathcal{R}(\xi) \neq \emptyset$. The subprincipal symbol of $P(t, D_t, D_x)$ is defined by

$$sub\ \sigma(P)(t, \tau, \xi) = P_{m-1}(t, \tau, \xi) + \frac{i}{2} \partial_t \partial_\tau p(t, \tau, \xi).$$

We assume

(L-1) for any $T > 0$ there is $C > 0$ such that

$$(1.2) \qquad \min\{\min_{s \in \mathcal{R}(\xi)} |t - s|, 1\} |sub\ \sigma(P)(t, \tau, \xi)| \leq C h_{m-1}(t, \tau, \xi)^{1/2}$$
$$\text{for } (t, \tau, \xi) \in [0, T] \times \mathbf{R} \times S^{n-1}$$

as the Levi condition for the $(m-1)$-th order terms of P. Here we define $\min_{s \in \mathcal{R}(\xi)} |t - s| = 1$ when $\mathcal{R}(\xi) = \emptyset$. To impose the Levi condition on the $(m-2)$-th order terms of P we have to define some quantities. Let $z^0 \equiv (t_0, \tau_0, \xi^0) \in [0, \infty) \times \mathbf{R} \times S^{n-1}$ satisfy $(\partial_\tau^k p)(z^0) = 0$ ($0 \leq k \leq 2$). Define a monic polynomial $p(t,\tau,\xi;z^0)$ of τ of degree 3 satisfying the following:

$$\begin{cases} p(t,\tau,\xi;z^0) \text{ is defined for } (t,\xi)\in\mathcal{U}(z^0) \text{ and } p(t,\tau,\xi) \text{ is divided} \\ \text{by } p(t,\tau,\xi;z^0) \text{ as polynomials of } \tau, \text{ and, putting } \tilde{p}(t,\tau,\xi;z^0) = \\ p(t,\tau,\xi)/p(t,\tau,\xi;z^0), \\ \quad \tau \in I(z^0) \quad \text{if } (t,\xi)\in\mathcal{U}(z^0),\ |\xi|=1 \text{ and } p(t,\tau,\xi;z^0)=0, \\ \quad \tilde{p}(t,\tau,\xi;z^0)\neq 0 \quad \text{if } (t,\xi)\in\mathcal{U}(z^0),\ |\xi|=1 \text{ and } \tau\in I(z^0), \end{cases}$$

where $\mathcal{U}(z^0)$ is a neighborhood of (t_0,ξ^0) and $I(z^0)$ is a neighborhood of τ_0. Then we write

$$p(t,\tau,\xi;z^0) = \tau^3 + a_1(t,\xi;z^0)\tau^2 + a_2(t,\xi;z^0)\tau + a_3(t,\xi;z^0).$$

We define

$$\begin{aligned} (1.3)\qquad Q(t,\tau,\xi;z^0) =& P_{m-2}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{1}{4}\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\partial_t\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{i}{12}\partial_\tau^2 sub\ \sigma(P)(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0) \\ &+ \frac{1}{24}(\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0))^2\cdot\partial_\tau\tilde{p}(t,\tau,\xi;z^0) \\ &\qquad\qquad \text{for } (t,\xi)\in\mathcal{U}(z^0) \text{ and } \tau\in\mathbf{R}. \end{aligned}$$

The Levi condition for the $(m-2)$-th order terms of P is the following:

(L-2) For any $z^0\in[0,\infty)\times\mathbf{R}\times S^{n-1}$ with $(\partial_\tau^k p)(z^0)=0$ ($0\le k\le 2$) there is $C>0$ such that

$$\begin{aligned} (1.4)\qquad &\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|^2,1\}|Q(t,-a_1(t,\xi;z^0)/3,\xi;z^0)| \\ &\le Ch_{m-2}(t,-a_1(t,\xi;z^0)/3,\xi)^{1/2} \\ &\qquad\qquad \text{for } (t,\xi)\in\mathcal{U}(z^0) \text{ with } |\xi|=1. \end{aligned}$$

We note that

$$\begin{aligned} (1.5)\qquad Q(t,\tau,\xi;z^0) =& P_1(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi) \\ &+ \frac{i}{12}\partial_\tau^2 P_2(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi) \end{aligned}$$

when $m=3$. In [11] we defined the sub-sub-principal symbol $sub^2\,\sigma(P)(t,\tau,\xi)$ of P by the right-hand side of (1.5).

Now we can state our main result.

THEOREM **1.2.** *We assume that the conditions* (A-1), (A-2), (H), (T), (L-1) *and* (L-2) *are satisfied. Then the Cauchy problem* (CP) *is* C^∞ *well-posed. Moreover,* (CP) *has finite propagation property, more precisely, if* $(t_0, x^0) \in (0, \infty) \times \mathbf{R}^n$ *and* $u \in C^\infty([0, \infty) \times \mathbf{R}^n)$ *satisfies* (CP), $u_j(x) = 0$ *near* $\{x \in \mathbf{R}^n;\ (0, x) \in K^-_{(t_0,x^0)}\}$ $(\ 0 \le j \le m-1)$ *and* $f = 0$ *near* $K^-_{(t_0,x^0)}$ *(in* $[0, \infty) \times \mathbf{R}^n$*), then* $(t_0, x^0) \notin \operatorname{supp} u$.

Assume that $m \ge 2$, and put

$$\mu_{j,k}(t, \xi) = (\lambda_j(t, \xi) - \lambda_k(t, \xi))^2, \quad M = \binom{m}{2}$$

and define $\{D_l(t, \xi)\}_{1 \le l \le M}$ by

$$\tau^M + \sum_{l=1}^{M} D_l(t, \xi)\tau^{M-l} = \prod_{1 \le j < k \le m} (\tau + \mu_{j,k}(t, \xi)).$$

Note that $D_M(t, \xi)$ $(\equiv D(t, \xi))$ is the discriminant of $p(t, \tau, \xi) = 0$ in τ. Putting $D_0(t, \xi) \equiv 1$, for each $\xi \in S^{n-1}$ there is $r(\xi) \in \mathbf{Z}_+$ such that $0 \le r(\xi) \le M$ and

$$\begin{aligned} &D_M(t, \xi) \equiv \cdots \equiv D_{M-r(\xi)+1}(t, \xi) \equiv 0 \text{ in } t, \\ &D_{M-r(\xi)}(t, \xi) \not\equiv 0 \text{ in } t. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &D_{M-r(\xi)}(t, \xi) = \prod_{\substack{1 \le j < k \le m \\ \mu_{j,k}(t,\xi) \not\equiv 0 \text{ in } t}} \mu_{j,k}(t, \xi), \\ &r(\xi) = \#\{(j, k);\ 1 \le j < k \le m \text{ and } \mu_{j,k}(t, \xi) \equiv 0 \text{ in } t\}. \end{aligned}$$

We define

$$\mathcal{R}_0(\xi) = \{(\operatorname{Re} \lambda)_+;\ \lambda \in \Omega \text{ and } D_{M-r(\xi)}(\lambda, \xi) = 0\} \quad \text{for } \xi \in S^{n-1},$$

where $a_+ = \max\{0, a\}$ for $a \in \mathbf{R}$. By Lemma 2.1 below we may assume that for any $T > 0$ there is $N_T \in \mathbf{Z}_+$ satisfying

$$\#(\mathcal{R}_0(\xi) \cap [0, T]) \le N_T \quad \text{for } \xi \in S^{n-1},$$

modifying Ω if necessary.

THEOREM **1.3.** *Assume that* $n \le 2$*, and that the condition* (T) *and the following conditions* (A) *and* (H)$'$ *are satisfied:*

(A) $a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j, j-1, j-2$) *are real analytic in* $[0,\infty)$.

(H)$'$ $p(t,\tau,\xi)$ *is hyerbolic with respect to* ϑ *for* $t \in [0,\infty)$.

If the Cauchy problem (CP) *is* C^∞ *well-posed and has finite propagation property, then for any compact interval* $I \subset (0,\infty)$ *the following conditions* (L-1)$_I$ *and* (L-2)$_I$ *are satisfied:*

(L-1)$_I$ *There is* $C > 0$ *such that*

$$\min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|, 1\} |sub\ \sigma(P)(t,\tau,\xi)| \le C h_{m-1}(t,\tau,\xi)^{1/2}$$
$$\text{for } (t,\tau,\xi) \in I \times \mathbf{R} \times S^{n-1}.$$

(L-2)$_I$ *For any* $z^0 = (t_0,\tau_0,\xi^0) \in I \times \mathbf{R} \times S^{n-1}$ *with* $(\partial_\tau^k p)(z^0) = 0$ ($0 \le k \le 2$), *there are* $\hat{\delta} > 0$, *a neighborhood* U *of* ξ^0 *and* $C > 0$ *such that*

$$\begin{aligned}&\min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^2, 1\} |Q(t, -a_1(t,\xi;z^0)/3, \xi; z^0)| \\ &\le C h_{m-2}(t, -a_1(t,\xi;z^0)/3, \xi)^{1/2} \\ &\qquad \text{for } (t,\xi) \in (I \cap [t_0-\hat{\delta}, t_0+\hat{\delta}]) \times (S^{n-1} \cap U).\end{aligned}$$

Let U be a semi-algebraic set in $\mathbf{R}$, and let $h(t)$ be a function defined in U. For the definition of semi-algebraic sets we refer to [14], for example. We say that $h(t)$ is semi-algebraic in U if the graph $\{(t,h(t)) \in \mathbf{R}^2;\ t \in U\}$ is a semi-algebraic set. For basic properties of semi-algebraic functions we refer to [14] and [15].

Theorem **1.4.** *Assume that the conditions* (H)$'$, (T) *and the following condition* (A)$'$ *are satisfied:*

(A)$'$ $a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j, j-1, j-2$) *are semi-algebraic in* $[0,\infty)$.

Then the conditions (L-1)$_{[0,T]}$ *and* (L-2)$_{[0,T]}$ *for any* $T > 0$ *are satisfied if the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property.*

The remainder of this paper is organized as follows. §2 and §3 will be divided into subsections. In §2 we shall prove Theorem 1.2. Theorems 1.3 and 1.4 will be proved in §3.

2. Proof of Theorem 1.2

2.1. Preliminaries

Let I be an interval of $\mathbf{R}$, and let Γ be an open cone or a closed cone in $\mathbf{R}^n \setminus \{0\}$. Here 'cone' means that its vertex is the origin. Let $\kappa, \kappa' \in \mathbf{R}$. We say that $a(t,\xi) \in S^{\kappa}_{1,0}(I \times \Gamma)$ if $a(t,\xi) \in C^\infty(I \times \Gamma)$ and

$$(2.1) \qquad |D_t^j \partial_\xi^\alpha a(t,\xi)| \le C_{j,\alpha} |\xi|^{\kappa - |\alpha|}$$
$$\text{for } (t,\xi) \in I \times (\Gamma \cap \{|\xi| \ge 1\}) \text{ and any } j \in \mathbf{Z}_+ \text{ and } \alpha \in (\mathbf{Z}_+)^n.$$

When $a(t,\xi;\varepsilon)$ also depends on a parameter ε, we say that $a(t,\xi;\varepsilon) \in S^{\kappa}_{1,0}(I \times \Gamma)$ uniformly in ε if the $C_{j,\alpha}$ in (2.1) with $a(t,\xi)$ replaced by $a(t,\xi;\varepsilon)$ can be chosen so that they do not depend on ε. Moreover, we say that $a(t,\tau,\xi) \in \mathcal{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$ if $a(t,\tau,\xi) = \sum_{j=0}^{[\kappa]} a_j(t,\xi)\tau^j$ and $a_j(t,\xi) \in S^{\kappa+\kappa'-j}_{1,0}(I \times \Gamma)$, where $[\kappa]$ denotes the largest integer $\le \kappa$ and $\mathcal{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma) = \{0\}$ if $\kappa < 0$. We also write $\mathcal{S}^{\kappa}_{1,0}(I \times \Gamma) = \mathcal{S}^{\kappa,0}_{1,0}(I \times \Gamma)$ and $\mathcal{S}^{\kappa,-\infty}_{1,0}(I \times \Gamma) = \bigcap_{\kappa' \in \mathbf{R}} \mathcal{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$. When $a(t,\tau,\xi;\varepsilon) = \sum_{j=0}^{[\kappa]} a_j(t,\xi;\varepsilon)\tau^j$ depend on a parameter ε, we say that $a(t,\tau,\xi;\varepsilon) \in \mathcal{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$ uniformly in ε if $a_j(t,\xi;\varepsilon) \in S^{\kappa+\kappa'-j}_{1,0}(I \times \Gamma)$ uniformly in ε.

LEMMA **2.1.** *Let Γ be a closed cone in $\mathbf{R}^n \setminus \{0\}$, and let $a(t,\xi)$ be a real analytic symbol defined in $[0,1] \times \Gamma$, which is positively homogeneous in ξ. So there is a complex neighborhood Ω of $[0,1]$ such that $a(t,\xi)$ is holomorphic in $t \in \Omega$ for $\xi \in \Gamma$. Put*

$$\mathcal{R}_a(\xi) = \begin{cases} \{\lambda \in \Omega;\ a(\lambda,\xi) = 0\} & \text{if } a(t,\xi) \not\equiv 0 \text{ in } t, \\ \emptyset & \text{if } a(t,\xi) \equiv 0 \text{ in } t \end{cases}$$

for $\xi \in \Gamma \cap S^{n-1}$. Then there are $N \in \mathbf{Z}_+$ and $C > 0$ such that $\#\mathcal{R}_a(\xi) \le N$ for $\xi \in \Gamma \cap S^{n-1}$ and

$$\min\{\min_{s \in \mathcal{R}_a(\xi)} |t-s|, 1\} |\partial_t a(t,\xi)| \le C|a(t,\xi)| \quad \text{for } (t,\xi) \in [0,1] \times (\Gamma \cap S^{n-1})$$

REMARK. It follows from the proof that there are $C_k > 0$ ($k \in \mathbf{N}$) satisfying

$$\min\{\min_{s \in \mathcal{R}_a(\xi)} |t-s|^k, 1\} |\partial_t^k a(t,\xi)| \le C_k |a(t,\xi)| \quad \text{if } 1 \le k \le N$$
$$\min\{\min_{s \in \mathcal{R}_a(\xi)} |t-s|^N, 1\} |\partial_t^k a(t,\xi)| \le C_k |a(t,\xi)| \quad \text{if } k > N$$

for $(t,\xi) \in [0,1] \times (\Gamma \cap S^{n-1})$.

Proof. Replacing $[-\delta, \delta]$ and $\overline{U}$ with $[0,1]$ and $\{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2\}$, respectively, we apply the arguments as in the proof of Lemma 2.2 of [12]. Put

$$\kappa(\xi) = \int_0^1 |a(t,\xi)|^2\, dt$$

If $\kappa(\xi) \equiv 0$, then the lemma become trivial. So we may assume that $\kappa(\xi) \not\equiv 0$. Let $\xi^0 \in \{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2\}$. We apply Hironaka's resolution theorem to $\kappa(\xi)$ (see [1]). Then there are an open neighborhood $U(\xi^0)$ of ξ^0, a real analytic manifold $\widetilde{U}(\xi^0)$, a proper analytic mapping $\varphi \equiv \varphi(\xi^0) : \widetilde{U}(\xi^0) \ni \tilde{u} \mapsto \varphi(\tilde{u})\,(\equiv \varphi(\tilde{u};\xi^0)) \in U(\xi^0)$ satisfying the following:

(i) φ: $\widetilde{U}(\xi^0) \setminus \widetilde{A} \to U(\xi^0) \setminus A$ is an isomorphism, where $A = \{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2$ and $\kappa(\xi) = 0\}$ and $\widetilde{A} = \varphi^{-1}(A)$.

(ii) For each $p \in \widetilde{U}(\xi^0)$ there are local analytic coordinates $X\,(\equiv X^p) = (X_1, \cdots, X_n)\,(= (X_1^p, \cdots, X_n^p))$ centered at p, $r(p) \in \mathbf{Z}_+$ with $r(p) \le n$, $s_k(p) \in \mathbf{N}$ ($1 \le k \le r(p)$), a neighborhood $\widetilde{U}(\xi^0;p)$ of p and a real analytic function $e(X)$ in $\widetilde{V}(\xi^0;p)$ such that $e(X) > 0$ for $X \in \widetilde{V}(\xi^0;p)$ and

$$\kappa(\varphi(\tilde{u})) = e(X(\tilde{u})) \prod_{k=1}^{r(p)} X_k(\tilde{u})^{2s_k(p)} \quad (\ \tilde{u} \in \widetilde{U}(\xi^0;p)),$$

where $\widetilde{V}(\xi^0;p) = \{X(\tilde{u});\ \tilde{u} \in \widetilde{U}(\xi^0;p)\}$ and $\prod_{k=1}^{r(p)} \cdots = 1$ if $r(p) = 0$.

Here $\widetilde{V}(\xi^0;p)$ is a neighborhood of 0 in $\mathbf{R}^n$ and we have used the fact that $\kappa(\xi) \ge 0$. Define $\tilde{\varphi}\,(\equiv \tilde{\varphi}(\xi^0,p)) : \widetilde{V}(\xi^0;p) \to U(\xi^0)$ by $\tilde{\varphi}(X(\tilde{u}))\,(\equiv \tilde{\varphi}(X^p(\tilde{u});\xi^0,p)) = \varphi(\tilde{u})\,(\equiv \varphi(\tilde{u};\xi^0))$ for $\tilde{u} \in \widetilde{U}(\xi^0;p)$. Let $U_0(\xi^0)$ be a compact neighborhood of ξ^0 in $U(\xi^0)$, and put $\widetilde{U}_0(\xi^0) = \varphi^{-1}(U_0(\xi^0))$. Fix $p \in \widetilde{U}_0(\xi^0)$, and put

$$\alpha(p) = (s_1(p), \cdots, s_{r(p)}(p), 0, \cdots, 0) \in (\mathbf{Z}_+)^n,$$
$$c_\alpha(t;p) = \frac{1}{\alpha!}\partial_X^\alpha a(t, \tilde{\varphi}(X))|_{X=0}.$$

Note that $\alpha \ge \alpha(p)$ if $\alpha \in (\mathbf{Z}_+)^n$ and $c_\alpha(t;p) \not\equiv 0$ in t (see the proof of Lemma 2.2 of [12]). So we can write

$$a(t, \tilde{\varphi}(X)) = X^{\alpha(p)} a(t, X; p),$$
$$a(t, X; p) = c_{\alpha(p)}(t;p) + b(t, X; p),$$

where $b(t,X;p)$ is real analytic in (t,X) and satisfies $b(t,0;p) = 0$. Since $c_{\alpha(p)}(t;p) \not\equiv 0$ in t, we can apply the Weierstrass preparation theorem to

$a(t, X; p)$ at $(t, X) = (t_0, 0)$, where $t_0 \in [0,1]$. Then there are $\delta(p, t_0) > 0$, a neighborhood $\widetilde{V}(p, t_0)$ of 0 in $\widetilde{V}(\xi^0; p)$, $m(p, t_0) \in \mathbf{Z}_+$, a real analytic function $c(t, X; p, t_0)$ defined in $[t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \widetilde{V}(p, t_0)$ and real analytic functions $a_k(X; p, t_0)$ defined in $\widetilde{V}(p, t_0)$ ($1 \le k \le m(p, t_0)$) such that $c(t, X; p, t_0) \neq 0$ and

$$\begin{aligned} &a(t, X; p) \\ &\quad = c(t, X; p, t_0)(t^{m(p,t_0)} + a_1(X; p, t_0) t^{m(p,t_0)-1} + \cdots + a_{m(p,t_0)}(X; p, t_0)) \end{aligned}$$

in $[t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \widetilde{V}(p, t_0)$. Note that $\delta(p, t_0)$, $\widetilde{V}(p, t_0)$, $m(p, t_0)$, $c(t, X; p, t_0)$ and the $a_k(X; p, t_0)$ also depend on ξ^0. So we can write

$$a(t, \tilde{\varphi}(X)) = X^{\alpha(p)} c(t, X; p, t_0) \prod_{k=1}^{m(p,t_0)} (t - t_k(X; p, t_0))$$

for $(t, X) \in [t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \widetilde{V}(p, t_0)$. We may assume that

$$\mathcal{R}_a(\tilde{\varphi}(X)) = \begin{cases} \{t_1(X; p, t_0), \cdots, t_{m(p,t_0)}(X; p, t_0)\} & \text{if } X_1 \cdots X_{r(p)} \neq 0, \\ \emptyset & \text{if } X_1 \cdots X_{r(p)} = 0. \end{cases}$$

Then we have

$$\min\{\min_{s \in \mathcal{R}_a(\tilde{\varphi}(X))} |t - s|, 1\} |\partial_t a(t, \tilde{\varphi}(X))| \le C(p, t_0) |a(t, \tilde{\varphi}(X))| \tag{2.2}$$

for $(t, X) \in [t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \widetilde{V}(p, t_0)$, where $C(p, t_0) > 0$. Since $[0,1] \times \{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2\}$ and $\widetilde{U}_0(\xi^0)$ are compact, compactness arguments prove the lemma. $\square$

From the assumption (T) there are $\delta_1 > 0$, $N_0 \in \mathbf{N}$, $m(j, k) \in \mathbf{N}$, open cones Γ_j in $\mathbf{R}^n \setminus \{0\}$, $r(j) \in \mathbf{N}$, compact intervals $J_{j,k}$ and $p^{j,k}(t, \tau, \xi) \in \mathcal{S}_{1,0}^{m(j,k)}([0, \delta_1] \times (\overline{\Gamma}_j \setminus \{0\}))$ ($1 \le j \le N_0$, $1 \le k \le r(j)$) such that $m(j, k) \le 3$, the $p^{j,k}(t, \tau, \xi)$ are monic polynomials of τ and positively homogeneous of degree $m(j, k)$ in $(\tau, \xi) \in \mathbf{R} \times (\overline{\Gamma}_j \setminus \{0\})$ such that $\bigcup_{j=1}^{N_0} \Gamma_j \supset S^{n-1}$, $J_{j,k} \cap J_{j,l} = \emptyset$ for $1 \le j \le N_0$ and $1 \le k < l \le r(j)$,

$$p(t, \tau, \xi) = \prod_{k=1}^{r(j)} p^{j,k}(t, \tau, \xi) \quad \text{for } (t, \tau, \xi) \in [0, \delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1}), \tag{2.3}$$

$\tau \in J_{j,k}$ if $1 \le j \le N_0$, $1 \le k \le r(j)$, $(t, \xi) \in [0, \delta_1] \times (\overline{\Gamma}_j \cap S^{n-1})$, $\tau \in \mathbf{C}$ and $p^{j,k}(t, \tau, \xi) = 0$, and for each (j, k) with $1 \le j \le N_0$ and $1 \le k \le r(j)$ there is $(\hat{\tau}, \hat{\xi}) \in \mathbf{R} \times (\Gamma_j \cap S^{n-1})$ satisfying

$$(\partial_\tau^\mu p^{j,k})(0, \hat{\tau}, \hat{\xi}) = 0 \quad (\ 0 \le \mu \le m(j, k) - 1).$$

Let $\delta > 0$ and Γ be a closed cone in $\mathbf{R}^n \setminus \{0\}$. We say that $a(t,\xi) \in \mathcal{A}_{cl}([0,\delta]\times\Gamma)$ if $a(t,\xi)$ is real analytic in $[0,\delta]\times\Gamma$ and a classical symbol, *i.e.*, when $a(t,\xi) \not\equiv 0$, there are $\kappa \in \mathbf{Z}$ and real analytic symbols $a_j(t,\xi)$ $(j \in \mathbf{Z}_+)$ such that $a_0(t,\xi) \not\equiv 0$, $a_j(t,\xi)$ is positively homogeneous of degree $(\kappa - j)$ in ξ $(\, j \in \mathbf{Z}_+)$ and $a(t,\xi) \sim \sum_{j=0}^{\infty} a_j(t,\xi)$, *i.e.*,

$$|a(t,\xi) - \sum_{j=0}^{N-1} a_j(t,\xi)| \le C_N |\xi|^{\kappa - N}$$

for $(t,\xi) \in [0,\delta]\times\Gamma$ with $|\xi| \ge 1$ and $N \in \mathbf{N}$, where $C_N > 0$. Here $a_0(t,\xi)$ is called the principal symbol of $a(t,\xi)$.

Lemma **2.2.** *Assume that $p(t,\tau,\xi) \in \mathcal{A}_{cl}([0,\delta]\times\Gamma)[\tau]$ is a monic polynomial of τ, positively homogeneous of degree m $(\in \mathbf{N})$ in (τ,ξ) and hyperbolic in τ, i.e., $p(t,\tau \pm i,\xi) \ne 0$ for $(t,\tau,\xi) \in [0,\delta]\times\mathbf{R}\times\Gamma$. Write*

$$p(t,\tau,\xi) = \prod_{j=1}^{m} (\tau - \lambda_j(t,\xi)).$$

Then, for each fixed $\xi \in \Gamma \cap S^{n-1}$ we can enumerate $\{\lambda_j(t,\xi)\}$ so that the $\lambda_j(t,\xi)$ are real analytic in $t \in [0,\delta]$. Moreover, for any $\nu \in \mathbf{Z}_+$ there is $\mathcal{N}_\nu (\equiv \mathcal{N}_\nu(p)) \subset \Gamma$ satisfying the following:

(i) *$\lambda\xi \in \mathcal{N}_\nu$ if $\lambda > 0$ and $\xi \in \mathcal{N}_\nu$.*

(ii) *$\mu_n(\mathcal{N}_\nu) = 0$.*

(iii) *There is $N_\nu \in \mathbf{Z}_+$ such that*

$$\begin{aligned}&\#\{t \in [0,\delta];\ \partial_t^\mu(\lambda_j(t,\xi) - \lambda_k(t,\xi)) = 0\} \le N_\nu \\ &\qquad \text{if } 0 \le \mu \le \nu,\ 1 \le j < k \le m \text{ and } \partial_t^\mu(\lambda_j(t,\xi) - \lambda_k(t,\xi)) \not\equiv 0 \text{ in } t, \\ &\#\{t \in [0,\delta];\ \partial_t^\mu \lambda_j(t,\xi) = 0\} \le N_\nu \\ &\qquad \text{if } 0 \le \mu \le \nu,\ 1 \le j \le m \text{ and } \partial_t^\mu \lambda_j(t,\xi) \not\equiv 0 \text{ in } t\end{aligned}$$

for $\xi \in \Gamma \setminus \mathcal{N}_\nu$.

Here μ_n denotes the Lebesgue measure in $\mathbf{R}^n$.

Remark. The lemma is a generalization of Lemma 2.3 in [11]. We also need to apply the lemma to $\partial_\tau p^{j,k}(t,\tau,\xi)$ with $m(j,k) = 3$.

PROOF. We will modify the proof of Lemma 2.3 of [11]. First fix $\xi \in \Gamma \cap S^{n-1}$. For $t_0 \in [0, \delta]$ $\mathcal{A}_{t_0}$ denotes the convergent power series ring of $(t - t_0)$. Since $\mathcal{A}_{t_0}$ is a unique factorization domain, $\mathcal{A}_{t_0}[\tau]$ is also a unique factorization domain. Applying the same argument as in the proof of Lemma 2.3 of [11], we can prove the first part of the lemma. In doing so we note that $\lambda_{j_0}(t_0 + z^r, \xi)$ is analytic in a complex neighborhood of $z = 0$ with some $r \in \mathbf{N}$ and that $\lambda_{j_0}(t_0 + z^r, \xi)$ can be expanded as a power series of z (see the proof of Lemma 2.3 of [11]). Hyperbolicity implies that $\lambda_{j_0}(t_0 + z^r, \xi)$ is real if z^r is real, and that $\lambda_{j_0}(t_0 + z^r, \xi)$ is a power series of z^r. So we can take $r = 1$ and $\lambda_{j_0}(t, \xi)$ is analytic in t near $t = t_0$. We denote by Σ the quotient field of $\mathcal{A}_{cl}([0, \delta] \times \Gamma)$. Then $\Sigma[\tau]$ is a unique factorization domain and $p(t, \tau, \xi) \in \Sigma[\tau]$. Write

$$\tau p(t, \tau, \xi) = p^1(t, \tau, \xi)^{r_1} \cdots p^\sigma(t, \tau, \xi)^{r_\sigma},$$

where $\sigma, r_j \in \mathbf{N}$, the $p^j(t, \tau, \xi)$ ($\in \Sigma[\tau]$) are irreducible in $\Sigma[\tau]$ and $p^j(t, \tau, \xi)$ and $p^k(t, \tau, \xi)$ are mutually prime if $j \neq k$. Define $q(t, \tau, \xi) = \prod_{j=1}^{\sigma} p^j(t, \tau, \xi)$, and let $D(t, \xi)$ be the discriminant of $q(t, \tau, \xi) = 0$ in τ. Then there are $d_k(t, \xi) \in \mathcal{A}_{cl}([0, \delta] \times \Gamma) \setminus \{0\}$ ($k = 0, 1$) such that

$$D(t, \xi) = d_0(t, \xi)/d_1(t, \xi),$$

since $D(t, \xi) \neq 0$ in Σ. Here we may assume that the $d_k(t, \xi)$ are positively homogeneous in ξ (see the proof of Lemma 2.3 of [11]). Write $q(t, \tau, \xi) = \tau^{\hat{m}} + \sum_{j=1}^{\hat{m}-1} \hat{a}_j(t, \xi) \tau^{\hat{m}-j}$. Similarly, there are $\hat{a}_j^l(t, \xi) \in \mathcal{A}_{cl}([0, \delta] \times \Gamma)$ ($1 \leq j \leq \hat{m} - 1$, $l = 0, 1$) such that the $\hat{a}_j^l(t, \xi)$ are positively homogeneous in ξ, $\hat{a}_j^1(t, \xi) \neq 0$ (in $\mathcal{A}_{cl}([0, \delta] \times \Gamma)$) and $\hat{a}_j(t, \xi) = \hat{a}_j^0(t, \xi)/\hat{a}_j^1(t, \xi)$, since the $\hat{a}_j(t, \xi)$ are positively homogeneous in ξ. Put

$$\mathcal{N}_0 = \{\xi \in \Gamma;\ d_0(t, \xi) d_1(t, \xi) \prod_{j=1}^{\hat{m}-1} \hat{a}_j^1(t, \xi) \equiv 0 \text{ in } t \in [0, \delta]\}.$$

Then we have $\mu_n(\mathcal{N}_0) = 0$. We can choose functions $\hat{\lambda}_j(t, \xi)$ ($1 \leq j \leq \hat{m}$) defined in $[0, \delta] \times (\Gamma \setminus \mathcal{N}_0)$ such that the $\hat{\lambda}_j(t, \xi)$ are real analytic in t for a fixed $\xi \in \Gamma \setminus \mathcal{N}_0$ and

$$q(t, \tau, \xi) = \prod_{j=1}^{\hat{m}} (\tau - \hat{\lambda}_j(t, \xi)) \quad \text{for } t \in [0, \delta] \text{ and } \xi \in \Gamma \setminus \mathcal{N}_0.$$

Note that

$$\{\hat{\lambda}_1(t, \xi), \cdots, \hat{\lambda}_{\hat{m}}(t, \xi)\} = \{0, \lambda_1(t, \xi), \cdots, \lambda_m(t, \xi)\}$$

for $(t,\xi) \in [0,\delta] \times (\Gamma \setminus \mathcal{N}_0)$. We may assume that $\hat{\lambda}_{\hat{m}}(t,\xi) \equiv 0$. Note that the $\hat{a}_j(t,\xi)$ are real analytic in $t \in [0,\delta]$ for $\xi \in \Gamma \setminus \mathcal{N}_0$. If $\xi \in \Gamma \setminus \mathcal{N}_0$ and

$$t \in D_\xi \equiv \{s \in [0,\delta];\ d_0(s,\xi) d_1(s,\xi) \prod_{j=1}^{\hat{m}-1} \hat{a}_j^1(s,\xi) \neq 0\},$$

then the roots of $q(t,\tau,\xi) = 0$ in τ are simple. It follows from Lemma 2.2 and its remark of [12] that there is $N_0 \in \mathbf{Z}_+$ satisfying $\#([0,\delta] \setminus D_\xi) \le N_0$ for $\xi \in \Gamma \setminus \mathcal{N}_0$. This proves the second part of the lemma for $\nu = 0$. Let $\xi \in \Gamma \setminus \mathcal{N}_0$. Then we have

$$\partial_\tau q(t,\tau,\xi)|_{\tau=\hat{\lambda}_j(t,\xi)} \cdot \partial_t \hat{\lambda}_j(t,\xi) + \partial_t q(t,\tau,\xi)|_{\tau=\hat{\lambda}_j(t,\xi)} = 0$$

for $1 \le j \le \hat{m}$. Therefore, for $t \in D_\xi$ we have

$$\partial_t \hat{\lambda}_j(t,\xi) = -\partial_t q(t,\tau,\xi)|_{\tau=\hat{\lambda}_j(t,\xi)} / \partial_\tau q(t,\tau,\xi)|_{\tau=\hat{\lambda}_j(t,\xi)} \quad (\ 1 \le j \le \hat{m}).$$

Since $\hat{\lambda}_{\hat{m}}(t,\xi) \equiv 0$, we have

$$\prod_{j=1}^{\hat{m}} \partial_t \hat{\lambda}_j(t,\xi) \equiv 0.$$

Noting that

$$\begin{aligned}
&\prod_{k\neq j} (\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) \partial_\tau q(t,\tau,\xi)|_{\tau=\hat{\lambda}_k(t,\xi)} \\
&= \prod_{1\le k,l\le \hat{m},\, k\neq l} (\hat{\lambda}_k(t,\xi) - \hat{\lambda}_l(t,\xi)) = (-1)^{\hat{m}(\hat{m}-1)/2} D(t,\xi)
\end{aligned}$$

for a fixed j with $1 \le j \le \hat{m}$, we can write the other fundamental symmetric expressions as follows:

$$\begin{aligned}
&\sum_{j=1}^{\hat{m}} \prod_{k\neq j} \partial_t \hat{\lambda}_k(t,\xi) \\
&= (-1)^{\hat{m}-1+\hat{m}(\hat{m}-1)/2} \\
&\quad \times \sum_{j=1}^{\hat{m}} \prod_{k\neq j} \{(\hat{\lambda}_k(t,\xi) - \hat{\lambda}_j(t,\xi)) \partial_t q(t,\tau,\xi)|_{\tau=\hat{\lambda}_k(t,\xi)}\} / D(t,\xi) \\
&= E_{\hat{m}-1}(t,\xi)/D(t,\xi), \\
&\quad \cdots
\end{aligned}$$

$$\sum_{j=1}^{\hat{m}} \partial_t \hat{\lambda}_j(t,\xi) = E_1(t,\xi)/D(t,\xi),$$

where the $E_k(t,\xi)$ are polynomials of $\{\partial_t^l \hat{a}_j(t,\xi)\}_{1\le j\le \hat{m}-1,\, l=0,1}$. Put

$$\begin{aligned}\tilde{p}(t,\tau,\xi) &= \tau^{\hat{m}} - E_1(t,\xi)D(t,\xi)^{-1}\tau^{\hat{m}-1} + E_2(t,\xi)D(t,\xi)^{-1}\tau^{\hat{m}-2} \\ &\quad + \cdots + (-1)^{\hat{m}-1}E_{\hat{m}-1}(t,\xi)D(t,\xi)^{-1}\tau \\ \Big(&= \prod_{j=1}^{\hat{m}} (\tau - \partial_t\hat{\lambda}_j(t,\xi))\Big).\end{aligned}$$

Let us repeat the above argument with τp replaced by $\tilde{p}$. We write

$$\tilde{p}(t,\tau,\xi) = \tilde{p}^1(t,\tau,\xi)^{r_1'} \cdots \tilde{p}^{\sigma'}(t,\tau,\xi)^{r_{\sigma'}'},$$

where $\sigma', r_j' \in \mathbf{N}$, the $\tilde{p}^j(t,\tau,\xi)$ ($\in \Sigma[\tau]$) are irreducible in $\Sigma[\tau]$ and $\tilde{p}^j(t,\tau,\xi)$ and $\tilde{p}^k(t,\tau,\xi)$ are mutually prime if $j \neq k$. Put

$$\tilde{q}(t,\tau,\xi) = \prod_{j=1}^{\sigma'} \tilde{p}^j(t,\tau,\xi),$$

and let $\widetilde{D}(t,\xi)$ be the discriminant of $\tilde{q}(t,\tau,\xi) = 0$ in τ. Then we can write

$$\widetilde{D}(t,\xi) = \tilde{d}_0(t,\xi)/\tilde{d}_1(t,\xi),$$

where $\tilde{d}_k(t,\xi) \in \mathcal{A}_{cl}([0,\delta]\times\Gamma)\setminus\{0\}$. Here we may assume that the $\tilde{d}_k(t,\xi)$ are positively homogeneous in ξ. Write

$$\begin{aligned}&\tilde{q}(t,\tau,\xi) = \tau^{\tilde{m}} + \sum_{j=1}^{\tilde{m}-1} \tilde{a}_j(t,\xi)\tau^{\tilde{m}-j}, \\ &\tilde{a}_j(t,\xi) = \tilde{a}_j^0(t,\xi)/\tilde{a}_j^1(t,\xi) \quad (\ 1 \le j \le \tilde{m}-1),\end{aligned}$$

where $\tilde{a}_j^l(t,\xi) \in \mathcal{A}_{cl}([0,\delta]\times\Gamma)$ ($1 \le j \le \tilde{m}-1$, $l = 0,1$) are positively homogeneous in ξ and $\tilde{a}_j^1(t,\xi) \neq 0$ in $\mathcal{A}_{cl}([0,\delta]\times\Gamma)$. Define

$$\mathcal{N}_1 = \{\xi \in \Gamma;\ \tilde{d}_0(t,\xi)\tilde{d}_1(t,\xi)\prod_{j=1}^{\tilde{m}-1}\tilde{a}_j^1(t,\xi) \equiv 0 \text{ in } t \in [0,\delta]\} \cup \mathcal{N}_0.$$

Then we have $\mu_n(\mathcal{N}_1) = 0$. It is obvious that $\mathcal{N}_1$ is a cone. Similarly, there is $N_1 \in \mathbf{Z}_+$ such that

$$\#\{t \in [0,\delta];\ \partial_t(\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) = 0\}$$

$$(\le \#\{t \in [0,\delta];\ \tilde{d}_0(t,\xi)\tilde{d}_1(t,\xi) \prod_{j=1}^{\tilde{m}-1} \tilde{a}_j^1(t,\xi) = 0\}) \le N_1$$

if $\xi \in \Gamma \setminus \mathcal{N}_1$, $1 \le j < k \le \hat{m}$ and $\partial_t(\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) \not\equiv 0$ in t. This proves the second part of the lemma for $\nu = 1$. Repeating the above arguments we can prove the lemma for $\nu = 2, 3, \cdots$, inductively. □

We choose $\rho(t) \in \mathcal{E}^{\{\kappa_0\}}(\mathbf{R})$ so that $\rho(t) \ge 0$, $\int \rho(t)\, dt = 1$ and $\operatorname{supp} \rho \subset \{t \in \mathbf{R};\ |t| \le 1\}$. Define

$$a_{j,\alpha}(t;\varepsilon) = \int \rho_\varepsilon(s) a_{j,\alpha}(t-s)\, ds \quad (3 \le j \le m,\ |\alpha| \le j-3),$$

$$P_k(t,\tau,\xi;\varepsilon) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t;\varepsilon)\tau^{m-j}\xi^\alpha \quad (0 \le k \le m-3),$$

$$P(t,\tau,\xi;\varepsilon) = \sum_{k=0}^{2} P_{m-k}(t,\tau,\xi) + \sum_{k=3}^{m} P_{m-k}(t,\tau,\xi;\varepsilon)$$

for $0 < \varepsilon \le 1$, where $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$.

We approximate $P(t,\tau,\xi)$ by $P(t,\tau,\xi;\varepsilon)$ in order to prove that (CP) has finite propagation property. We factorized $p(t,\tau,\xi)$ as (2.3). By the factorization theorem we can write

$$\text{(2.4)}\quad \begin{aligned} &P(t,\tau,\xi;\varepsilon) \\ &\quad = P^{j,1}(t,\tau,\xi;\varepsilon) \circ P^{j,2}(t,\tau,\xi;\varepsilon) \circ \cdots \circ P^{j,r(j)}(t,\tau,\xi;\varepsilon) + R_j(t,\tau,\xi;\varepsilon) \end{aligned}$$

for $1 \le j \le N_0$, $(t,\xi) \in [0,\delta_1] \times \overline{\Gamma}_j$ with $|\xi| \ge 1$ and $\varepsilon \in (0,1]$, where

$$P^{j,k}(t,\tau,\xi;\varepsilon) = p^{j,k}(t,\tau,\xi) + q_0^{j,k}(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon),$$

$q_l^{j,k}(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m(j,k)-1,-l}([0,\delta_1] \times (\overline{\Gamma}_j \setminus \{0\}))$ ($l = 0,1$) are positively homogeneous of degree $(m(j,k)-1-l)$ in (τ,ξ) for $|\xi| \ge 1$, $r^{j,k}(t,\tau,\xi;\varepsilon) \in \mathcal{S}_{1,0}^{m(j,k)-1,-2}([0,\delta_1] \times (\overline{\Gamma}_j \setminus \{0\}))$ uniformly in ε and $R_j(t,\tau,\xi;\varepsilon) \in \mathcal{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times (\overline{\Gamma}_j \setminus \{0\}))$ uniformly in ε (see, *e.g.*, [5] and, also, [12]). Here we denote by $a(t,\tau,\xi) \circ b(t,\tau,\xi)$ the symbol of $a(t,D_t,D_x)b(t,D_t,D_x)$. There are compact intervals $I_{j,k}$ ($1 \le j \le N_0$, $1 \le k \le r(j)$) and $M > 0$ such that

$$\bigcup_{k=1}^{r(j)} I_{j,k} = [-M,M], \quad \overset{\circ}{I}_{j,k} \cap \overset{\circ}{I}_{j,l} = \emptyset\ (1 \le j \le N_0,\ k \ne l),$$

$$\tau \in \overset{\circ}{I}_{j,k} \quad \text{if } 1 \le j \le N_0,\ 1 \le k \le r(j),\ (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1})$$

and $p^{j,k}(t,\tau,\xi)=0$,

where $\overset{\circ}{I}$ denotes the interior of I ($\subset \mathbf{R}$). For $1\le j\le N_0$ and $J\subset \{1,2,\cdots,r(j)\}$ we define

$$\Pi^j_J(t,\tau,\xi)=\prod_{1\le\mu\le r(j),\,\mu\notin J} p^{j,\mu}(t,\tau,\xi).$$

Now we fix j with $1\le j\le N_0$. Until the end of the proof of Lemma 2.4 except the statements of Lemmas 2.3 and 2.4 we omit the subscript j and the superscript j of Γ_j, $P^{j,k}(\cdot)$, $R_j(\cdot)$, $p^{j,k}(\cdot)$, $I_{j,k}$, $\Pi^j_J(t,\tau,\xi)$, $\cdots$, and "j" of $r(j)$, $m(j,k)$, $\cdots$ and so on, *i.e.*, we write Γ_j, $P^{j,k}(\cdot)$, $R_j(\cdot)$, $p^{j,k}(\cdot)$, $I_{j,k}$, $\Pi^j_J(t,\tau,\xi)$, $r(j)$, $m(j,k)$, $\cdots$ as Γ, $P^k(\cdot)$, $R(\cdot)$, $p^k(\cdot)$, I_k, $\Pi_J(t,\tau,\xi)$, r, $m(k)$, $\cdots$, respectively. Let $a(t,\tau,\xi)$ and $b(t,\tau,\xi)$ be defined in $\mathcal{U}$. We write

$$a(t,\tau,\xi)=O(b(t,\tau,\xi))\quad \text{for } (t,\tau,\xi)\in\mathcal{U}$$

if there is $C>0$ satisfying

$$|a(t,\tau,\xi)|\le C|b(t,\tau,\xi)|\quad \text{for } (t,\tau,\xi)\in\mathcal{U}.$$

Assume that $a(t,\tau,\xi),b(t,\tau,\xi)\ge 0$. We write

$$a(t,\tau,\xi)\approx b(t,\tau,\xi)\quad \text{for } (t,\tau,\xi)\in\mathcal{U}$$

if there is $C>0$ satisfying

$$C^{-1}a(t,\tau,\xi)\le b(t,\tau,\xi)\le Ca(t,\tau,\xi)\quad \text{for } (t,\tau,\xi)\in\mathcal{U}.$$

LEMMA **2.3.** *Let $1\le j\le N_0$. We have*

$$\begin{aligned}(2.5)\qquad sub\ \sigma(P)(t,\tau,\xi)&=\sum_{k=1}^{r(j)} sub\ \sigma(P^{j,k})(t,\tau,\xi)\Pi^j_{\{k\}}(t,\tau,\xi)\\ &\quad-\frac{i}{2}\sum_{1\le k<l\le r(j)}\{p^{j,k},p^{j,l}\}(t,\tau,\xi)\Pi^j_{\{k,l\}}(t,\tau,\xi)\\ &=\sum_{k=1}^{r(j)} sub\ \sigma(P^{j,k})(t,\tau,\xi)\Pi^j_{\{k\}}(t,\tau,\xi)+O(h_{m-1}(t,\tau,\xi)^{1/2})\\ &\qquad\qquad \textit{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times\overline{\Gamma}_j \textit{ with } |\xi|\ge 1,\end{aligned}$$

where

$$\{p^{j,k},p^{j,l}\}(t,\tau,\xi)=\partial_\tau p^{j,k}(t,\tau,\xi)\cdot\partial_t p^{j,l}(t,\tau,\xi)-\partial_t p^{j,k}(t,\tau,\xi)\partial_\tau p^{j,l}(t,\tau,\xi).$$

Moreover, we have

$$
\begin{aligned}
(2.6)\quad & P_{m-2}(t,\tau,\xi)\\
&= \sum_{k=1}^{r(j)} q_1^{j,k}\Pi^j_{\{k\}} - i \sum_{1\le k<l\le r(j),\,\nu\ne k,l} \partial_\tau p^{j,k}\cdot \partial_t p^{j,l}\cdot sub\ \sigma(P^{j,\nu})\Pi^j_{\{k,l,\nu\}}\\
&\quad + \sum_{1\le k<l\le r(j)} sub\ \sigma(P^{j,k}) sub\ \sigma(P^{j,l})\Pi^j_{\{k,l\}}\\
&\quad - \frac{i}{2}\sum_{1\le k,l\le r(j),\,k\ne l} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,l})\Pi^j_{\{k,l\}}\\
&\quad - i\sum_{1\le k<l\le r(j)} \{\partial_\tau p^{j,k}\cdot \partial_t sub\ \sigma(P^{j,l}) + \partial_t p^{j,l}\cdot \partial_\tau sub\ \sigma(P^{j,k})\}\Pi^j_{\{k,l\}}\\
&\quad - \frac{1}{2}\sum_{1\le k<l\le r(j)} \{\partial_\tau p^{j,k}\cdot \partial_t^2\partial_\tau p^{j,l} + \partial_t p^{j,l}\cdot \partial_t\partial_\tau^2 p^{j,k}\}\Pi^j_{\{k,l\}}\\
&\quad + O(h_{m-2}(t,\tau,\xi)^{1/2}) \quad \textit{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times\overline{\Gamma}_j \textit{ with } |\xi|\ge 1,
\end{aligned}
$$

where $q^{j,k} = q^{j,k}(t,\tau,\xi)$, $\Pi^j_{\{k\}} = \Pi^j_{\{k\}}(t,\tau,\xi)$, $\partial_\tau p^{j,k} = \partial_\tau p^{j,k}(t,\tau,\xi)$, *sub* σ $(P^{j,\nu}) =$ *sub* $\sigma(P^{j,\nu})(t,\tau,\xi)$, $\cdots$.

Proof. We can prove by induction on r that

$$
\begin{aligned}
(2.7)\qquad & P^1(t,\tau,\xi;\varepsilon)\circ P^2(t,\tau,\xi;\varepsilon)\circ\cdots\circ P^r(t,\tau,\xi;\varepsilon)\\
&- \Big[\prod_{k=1}^r p^k(t,\tau,\xi) + \sum_{k=1}^r q_0^k(t,\tau,\xi)\Pi_{\{k\}}\\
&\quad - i\sum_{1\le k<l\le r} \partial_\tau p^k\cdot\partial_t p^l\cdot \Pi_{\{k,l\}}\\
&\quad - i\sum_{1\le k<l\le r,\,\nu\ne k,l} \partial_\tau p^k\cdot\partial_t p^l\cdot q_0^\nu\Pi_{\{k,l,\nu\}}\\
&\quad + \sum_{k=1}^r q_1^k\Pi_{\{k\}} + \sum_{1\le k<l\le r} q_0^k q_0^l\Pi_{\{k,l\}}\\
&\quad - i\sum_{1\le k<l\le r} \{\partial_\tau p^k\cdot\partial_t q_0^l + \partial_t p^l\cdot\partial_\tau q_0^k\}\Pi_{\{k,l\}}\\
&\quad - \frac{1}{2}\sum_{1\le k<l\le r} \partial_\tau^2 p^k\cdot\partial_t^2 p^l\cdot\Pi_{\{k,l\}}\\
&\quad - \sum_{1\le k<l<\nu\le r} \partial_\tau^2 p^k\cdot\partial_t p^l\cdot\partial_t p^\nu\cdot\Pi_{\{k,l,\nu\}}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{1\le k<l<\nu\le r} \partial_\tau p^k\cdot\{\partial_t\partial_\tau p^l\cdot\partial_t p^\nu+\partial_\tau p^l\cdot\partial_t^2 p^\nu\}\Pi_{\{k,l,\nu\}} \\
&- \sum_{\substack{1\le k<l\le r,\, k<\nu<\mu\le r\\ \nu\ne l,\,\mu\ne l}} \partial_\tau p^k\cdot\partial_t p^l\cdot\partial_\tau p^\nu\cdot\partial_t p^\mu\cdot\Pi_{\{k,l,\nu,\mu\}}\Big] \\
&\in \mathcal{S}^{m-1,-2}_{1,0}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))\quad \text{uniformly in } \varepsilon,
\end{aligned}
$$

where, for example, $\Pi_{\{k\}}=\prod_{1\le l\le r,\, l\ne k} p^l(t,\tau,\xi)$. It follows from (1.1) that

$$
\begin{aligned}
&(2.8)\qquad h_{m(k)-l}(t,\tau,\xi;p^k)\approx h_{m-l}(t,\tau,\xi)\\
&\qquad\quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_k\times(\overline{\Gamma}\cap S^{n-1})\\
&\qquad\quad \text{if } 1\le k\le r \text{ and } 0\le l\le m(k),\\
&(2.9)\qquad h_{m(k)-l}(t,\tau,\xi;p^k)\approx(|\tau|+1)^{2m(k)-2l}\\
&\qquad\quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times(\mathbf{R}\setminus I_k)\times(\overline{\Gamma}\cap S^{n-1})\\
&\qquad\quad \text{if } 1\le k\le r \text{ and } 0\le l\le m(k),\\
&(2.10)\qquad h_{m-l}(t,\tau,\xi)\approx(|\tau|+1)^{2m-2l}\\
&\qquad\quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times((-\infty,-M)\cup(M,\infty))\times(\overline{\Gamma}\cap S^{n-1})\\
&\qquad\quad \text{if } 1\le l\le m.
\end{aligned}
$$

We have also, with $C>0$,

$$
(2.11)\qquad |\partial_t^\mu\partial_\tau^\nu p^k(t,\tau,\xi)|\le C h_{m(k)-\mu-\nu}(t,\tau,\xi)^{1/2}(|\tau|+1)^\mu
$$

for $1\le k\le r$, $\mu,\nu\in\mathbf{Z}_+$ with $\mu+\nu<m(k)$ and $(t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\overline{\Gamma}\cap S^{n-1})$. (2.7) – (2.11) prove the lemma. □

Now assume that (L-1) is satisfied. Let $1\le k_0\le r$ with $m(k_0)=3$. Then there is $C>0$ such that

$$
\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|sub\ \sigma(P^{k_0})(t,\tau,\xi)|\le C h_2(t,\tau,\xi;p^{k_0})^{1/2}
$$

for $(t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\overline{\Gamma}\cap S^{n-1})$. Write

$$
\begin{aligned}
p^k(t,\tau,\xi)&=\prod_{l=1}^{m(k)}(\tau-\lambda_l^k(t,\xi)),\\
p^k_\mu(t,\tau,\xi)&=p^k(t,\tau,\xi)/(\tau-\lambda^k_\mu(t,\xi))
\end{aligned}
$$

($1\le k\le r$, $1\le\mu\le m(k)$). Note that $h_2(t,\tau,\xi;p^{k_0})=\sum_{\mu=1}^3 p^{k_0}_\mu(t,\tau,\xi)^2$. It follows from Lagrange's interpolation formula that there are functions $b_\mu(t,\xi)$

($1 \le \mu \le 3$) and $C > 0$ satisfying

$$\text{(2.12)} \qquad sub\ \sigma(P^{k_0})(t,\tau,\xi) = \sum_{\mu=1}^{3} b_\mu(t,\xi) p_\mu^{k_0}(t,\tau,\xi),$$

$$\text{(2.13)} \qquad \min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|b_\mu(t,\xi)| \le C$$

for $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma} \cap S^{n-1})$ (see the proof of Lemma 2.5 of [11]).

Lemma **2.4.** *Assume that* (L-1) *is satisfied, and that* $1 \le j \le N_0$*,* $1 \le k_0 \le r(j)$ *and* $m(j,k_0) = 3$*. Then there is* $C > 0$ *such that*

$$\text{(2.14)} \qquad \min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|, 1\}|\partial_\tau^\mu sub\ \sigma(P^{j,k_0})(t,\tau,\xi)| \le C h_{2-\mu}(t,\tau,\xi;p^{j,k_0})^{1/2} \quad (\ \mu \le 2),$$

$$\text{(2.15)} \qquad \min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|^2, 1\}|\partial_t sub\ \sigma(P^{j,k_0})(t,\tau,\xi)|/(|\tau|+|\xi|) \le C h_1(t,\tau,\xi;p^{j,k_0})^{1/2}$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j$ *with* $|\xi| \ge 1$*, modifying* $\mathcal{R}(\xi)$ *if necessary.*

Proof. (2.14) easily follows from (2.12) and (2.13). Write $p^{k_0}(t,\tau,\xi) = \tau^3 + a_1^{k_0}(t,\xi)\tau^2 + a_2^{k_0}(t,\xi)\tau + a_3^{k_0}(t,\xi)$. We have

$$\begin{aligned} sub\ \sigma(P^{k_0})(t,\tau,\xi) &= sub\ \sigma(P^{k_0})(t,-a_1^{k_0}(t,\xi)/3,\xi) \\ &\quad + (\tau + a_1^{k_0}(t,\xi)/3)(\partial_\tau sub\ \sigma(P^{k_0}))(t,-a_1^{k_0}(t,\xi)/3,\xi) \\ &\quad + \frac{1}{2}(\tau + a_1^{k_0}(t,\xi)/3)^2(\partial_\tau^2 sub\ \sigma(P^{k_0}))(t,0,\xi), \end{aligned}$$

noting that $\deg_\tau sub\ \sigma(P^{k_0})(t,\tau,\xi) \le 2$. Therefore, we have

$$\text{(2.16)} \qquad \begin{aligned} |\partial_t sub\ \sigma(P^{k_0})(t,\tau,\xi)| &\le |\partial_t sub\ \sigma(P^{k_0})(t,-a_1^{k_0}/3,\xi)| \\ &\quad + \frac{1}{3}|\partial_t a_1^{k_0}(t,\xi)| \cdot |(\partial_\tau sub\ \sigma(P^{k_0}))(t,-a_1^{k_0}/3,\xi)| \\ &\quad + |\tau + a_1^{k_0}/3| \cdot |\partial_t(\partial_\tau sub\ \sigma(P^{k_0}))(t,-a_1^{k_0}/3,\xi)| \\ &\quad + |\tau + a_1^{k_0}/3|\, |\partial_t a_1^{k_0}(t,\xi)|/3 \cdot |(\partial_\tau^2 sub\ \sigma(P^{k_0}))(t,0,\xi)| \\ &\quad + \frac{1}{2}|\tau + a_1^{k_0}/3|^2 \cdot |\partial_t(\partial_\tau^2 sub\ \sigma(P^{k_0}))(t,0,\xi)|. \end{aligned}$$

Modifying $\mathcal{R}(\xi)$ if necessary, we can assume that

$$\{\mathrm{Re}\,\lambda;\ \lambda \in \Omega_1 \text{ and } sub\ \sigma(P^{k_0})(\lambda,-a_1^{k_0}(\lambda,\xi)/3,\xi) = 0\}$$

$$\subset \mathcal{R}(\xi) \quad \text{If } sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi) \not\equiv 0 \text{ in } t$$

for $\xi \in \overline{\Gamma} \cap S^{n-1}$, where Ω_1 is a compact complex neighborhood of $[0, \delta_1]$. Lemma 2.1 yields

$$\begin{aligned}
&\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|^2, 1\}|\partial_t sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)| \\
&\le C \min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|, 1\}|sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)|
\end{aligned}$$

for $(t, \xi) \in [0, \delta_1] \times \overline{\Gamma}$ with $|\xi| \ge 1$, where $C > 0$. Note that $-a_1^{k_0}(t,\xi)/3 \in I_{k_0}$ for $(t, \xi) \in [0, \delta_1] \times (\overline{\Gamma} \cap S^{n-1})$. So by (L-1), (2.8) and (1.1) we have, with $C > 0$,

$$\begin{aligned}
&\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|^2, 1\}|\partial_t sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)|/|\xi| \\
&\le C h_1(t, -a_1^{k_0}(t,\xi)/3, \xi; p^{k_0})^{1/2}
\end{aligned} \tag{2.17}$$

for $(t, \xi) \in [0, \delta_1] \times \overline{\Gamma}$ with $|\xi| \ge 1$. Since

$$|\tau + a_1^{k_0}(t,\xi)/3| \le \frac{1}{3}\sum_{\mu=1}^{3} |\tau - \lambda_\mu^{k_0}(t,\xi)| \le h_1(t,\tau,\xi;p^{k_0})^{1/2},$$

$$\begin{aligned}
&h_1(t, -a_1^{k_0}(t,\xi)/3, \xi; p^{k_0})^{1/2} \le \sum_{\mu=1}^{3} |a_1^{k_0}(t,\xi)/3 + \lambda_\mu^{k_0}(t,\xi)| \\
&\le \frac{2}{3}\{|\lambda_1^{k_0}(t,\xi) - \lambda_2^{k_0}(t,\xi)| + |\lambda_2^{k_0} - \lambda_3^{k_0}| + |\lambda_3^{k_0} - \lambda_1^{k_0}|\} \\
&\le \frac{4}{3}\sum_{\mu=1}^{3} |\tau - \lambda_\mu^{k_0}(t,\xi)| \le 4h_1(t,\tau,\xi;p^{k_0})^{1/2}
\end{aligned}$$

for $(t, \tau, \xi) \in [0, \delta_1] \times \mathbf{R} \times \overline{\Gamma}$ with $|\xi| \ge 1$, (2.14), (2.16) and (2.17) give (2.15). □

We wrote

$$p^{j,k}(t,\tau,\xi) = \tau^3 + a_1^{j,k}(t,\xi)\tau^2 + a_2^{j,k}(t,\xi)\tau + a_3^{j,k}(t,\xi)$$

if $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 3$. We say that (L-1) for $[0, \delta]$ is satisfied if (1.2) is satisfied with $[0, T]$ replaced by $[0, \delta]$, and that (L-2) for $[0, \delta]$ is satisfied if (1.4) is satisfied with $[0, \infty)$ replaced by $[0, \delta]$.

Lemma **2.5.** (i) (L-1) *for* $[0, \delta_1]$ *is satisfied if and only if there is* $C > 0$ *such that*

$$\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|sub\ \sigma(P^{j,k})(t,\tau,\xi)| \tag{2.18}$$

$$\le Ch_{m(j,k)-1}(t,\tau,\xi;p^{j,k})^{1/2} \quad \textit{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma} \cap S^{n-1})$$

provided $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 2$ *or* 3.
(ii) *Assume that* (L-1) *for* $[0,\delta_1]$ *is satisfied. Then* (L-2) *for* $[0,\delta_1]$ *is satisfied if and only if there is* $C > 0$ *such that*

$$(2.19)\quad \min\{\min_{s\in\mathcal{R}(\xi)} |t-s|^2, 1\}|sub^2\, \sigma(P^{j,k})(t, -a_1^{j,k}(t,\xi)/3,\xi)|$$
$$\le Ch_1(t, -a_1^{j,k}(t,\xi)/3,\xi;p^{j,k})^{1/2} \quad \textit{for } (t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \cap S^{n-1})$$

provided $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 3$, *modifying* $\mathcal{R}(\xi)$ *if necessary, where*

$$(2.20)\quad sub^2\, \sigma(P^{j,k})(t,\tau,\xi)$$
$$= q_1^{j,k}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi) + \frac{i}{12}\partial_\tau^2 q_0^{j,k}(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi).$$

Remark. In the lemma the interval $[0,\delta_1]$ can be replaced by a closed subinterval of $[0,\delta_1]$. From (2.5) we can see that whether the $sub\, \sigma(P^{j,k})(t,\tau,\xi)$ satisfy (2.18) or not does not depend on the order of the product in (2.4) while they depend on the order. Moreover, (2.26) below implies that whether the $sub^2\, \sigma(P^{j,k})(t,\tau,\xi)$ satisfy (2.19) or not does not depend on the order of the product in (2.4) while they depend on the order.

Proof. (2.5) and (2.8) – (2.10) prove the first assertion (i). Assume that $1 \le j \le N_0$, $z^0 = (t_0,\tau_0,\xi^0) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1})$, $(\partial_\tau^l p)(z^0) = 0$ ($0 \le l \le 2$), $1 \le k_0 \le r(j)$, $m(j,k_0) = 3$ and $\tau_0 \in \overset{\circ}{I}_{j,k_0}$. Note that $p(t,\tau,\xi;z^0) = p^{j,k_0}(t,\tau,\xi)$ and $\tilde{p}(t,\tau,\xi;z^0) = \Pi^j_{\{k_0\}}(t,\tau,\xi)$. Moreover, we may assume that $\mathcal{U}(z^0) = [0,\delta_1] \times (\overline{\Gamma}_j \setminus \{0\})$ and $I(z^0) = I_{j,k_0}$ in the definition of $Q(t,\tau,\xi;z^0)$, and $Q(t,\tau,\xi;z^0)$ is defined in $[0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \setminus \{0\})$. We say that

$$a(t,\tau,\xi) \equiv 0 \quad (\text{ mod (L-2)) at } z^0 \quad \text{for } (t,\xi) \in [0,\delta_1] \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1$$

if there is $C > 0$ such that

$$\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|^2, 1\}a(t, -a_1^{j,k_0}(t,\xi)/3,\xi)| \le Ch_{m-2}(t, -a_1^{j,k_0}(t,\xi)/3,\xi)^{1/2}$$
$$\text{for } (t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \cap S^{n-1}).$$

(L-2) implies that $Q(t,\tau,\xi;z^0) \equiv 0$ (mod (L-2)) at z^0 for $(t,\xi) \in [0,\delta_1] \times \overline{\Gamma}_j$ with $|\xi| \ge 1$. It follows from (2.6) that

$$(2.21)\quad q_1^{j,k_0}(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) = P_{m-2}(t,\tau,\xi)$$

$$
\begin{aligned}
&+ i \sum_{\substack{1\le k<l\le r(j)\\ k,l\ne k_0}} \partial_\tau p^{j,k}(t,\tau,\xi)\cdot \partial_t p^{j,l}(t,\tau,\xi)\cdot sub\ \sigma(P^{j,k_0})(t,\tau,\xi)\\
&\qquad\qquad \times \Pi^j_{\{k_0,k,l\}}(t,\tau,\xi)\\
&- \sum_{1\le k\le r(j),\, k\ne k_0} sub\ \sigma(P^{j,k_0}) sub\ \sigma(P^{j,k})\Pi^j_{\{k_0,k\}}\\
&+ \frac{i}{2} \sum_{1\le k\le r(j),\, k\ne k_0} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,k_0})\Pi^j_{\{k_0,k\}}\\
&+ i \sum_{k_0<k\le r(j)} \partial_t p^{j,k}\cdot \partial_\tau sub\ \sigma(P^{j,k_0})\cdot \Pi^j_{\{k_0,k\}}\\
&+ i \sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot \partial_t sub\ \sigma(P^{j,k_0})\cdot \Pi^j_{\{k_0,k\}}\\
&+ \frac{1}{2} \sum_{k_0<k\le r(j)} \partial_t p^{j,k}\cdot \partial_t\partial_\tau^2 p^{j,k_0}\cdot \Pi^j_{\{k_0,k\}}\\
&+ \frac{1}{2} \sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot \partial_t^2\partial_\tau p^{j,k_0}\cdot \Pi^j_{\{k_0,k\}} + O(h_{m-2}(t,\tau,\xi)^{1/2})\\
&\quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1,
\end{aligned}
$$

where $sub\ \sigma(P^{j,k_0}) = sub\ \sigma(P^{j,k_0})(t,\tau,\xi)$, $\cdots$, since

$$
\begin{aligned}
&\Pi^j_{\{k\}}(t,\tau,\xi) = O(p^{j,k_0}(t,\tau,\xi)|\xi|^{m-m(j,k)-3})\\
&= O(h_{m-2}(t,\tau,\xi)^{1/2}|\xi|^{-m(j,k)+2}) \quad (\ k\ne k_0),\\
&\partial_\tau p^{j,k_0}(t,\tau,\xi) = O(h_2(t,\tau,\xi;p^{j,k_0})^{1/2}),\\
&\partial_t p^{j,k_0}(t,\tau,\xi) = O(h_2(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|),\\
&\partial_t\partial_\tau p^{j,k_0}(t,\tau,\xi) = O(h_1(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|)\\
&\qquad\qquad\qquad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\overline{\Gamma} \text{ with } |\xi|\ge 1.
\end{aligned}
$$

It also follows from (1.3) and Lemma 2.3 that

$$
\begin{aligned}
(2.22)\quad & P_{m-2}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot \Pi^j_{\{k_0\}}(t,\tau,\xi)\\
&+ \frac{i}{12}\partial_\tau^2 sub\ \sigma(P)(t,\tau,\xi)\cdot \partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\\
&= Q(t,\tau,\xi;z^0) - \frac{1}{4}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot \partial_t\Pi^j_{\{k_0\}}(t,\tau,\xi)\\
&\quad - \frac{1}{24}(\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi))^2\cdot \partial_\tau \Pi^j_{\{k_0\}}(t,\tau,\xi),\\
(2.23)\quad & \{\partial_\tau^2 q_0^{j,k_0}(t,\tau,\xi)\cdot \Pi^j_{\{k_0\}}(t,\tau,\xi) - \partial_\tau^2 sub\ \sigma(P)(t,\tau,\xi)\}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)
\end{aligned}
$$

$$
\begin{aligned}
&= -\{2\partial_\tau sub\ \sigma(P^{j,k_0}) \cdot \partial_\tau \Pi^j_{\{k_0\}}(t,\tau,\xi) \\
&\qquad + sub\ \sigma(P^{j,k_0}) \cdot \partial_\tau^2 \Pi^j_{\{k_0\}}(t,\tau,\xi)\}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi) \\
&\quad + \frac{i}{2}\sum_{k_0<k\le r(j)} \{6\partial_t p^{j,k} - \partial_t\partial_\tau^2 p^{j,k_0}\cdot\partial_\tau p^{j,k}\}\Pi^j_{\{k_0,k\}}\cdot\partial_t\partial_\tau^2 p^{j,k_0} \\
&\quad - \frac{i}{2}\sum_{1\le k<k_0} \{6\partial_t p^{j,k} - \partial_t\partial_\tau^2 p^{j,k_0}\cdot\partial_\tau p^{j,k}\}\Pi^j_{\{k_0,k\}}\cdot\partial_t\partial_\tau^2 p^{j,k_0} \\
&\quad + O(h_{m-2}(t,\tau,\xi)^{1/2}) \\
&\qquad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1,
\end{aligned}
$$

since

$$
\begin{aligned}
&\partial_\tau^2 sub\ \sigma(P^{j,k_0})(t,\tau,\xi) = \partial_\tau^2 q_0^{j,k_0}(t,\tau,\xi), \\
&\partial_\tau^2\{p^{j,k_0}(t,\tau,\xi), p^{j,k}(t,\tau,\xi)\} \\
&= 6\partial_t p^{j,k}(t,\tau,\xi) - \partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot\partial_\tau p^{j,k}(t,\tau,\xi) \\
&\quad + O(h_1(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|^{m(j,k)-1}) \\
&\qquad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1 \text{ and } k\ne k_0, \\
&h_1(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|^{m-3} = O(h_{m-2}(t,\tau,\xi)^{1/2}) \\
&\qquad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1.
\end{aligned}
$$

Therefore, (2.20) – (2.23) yield

$$
\begin{aligned}
(2.24)\quad & sub^2\,\sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) \\
&= q_1^{j,k_0}(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) \\
&\quad + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot\Pi^j_{\{k_0\}} + \frac{i}{12}\partial_\tau^2 sub\ \sigma(P)\partial_t\partial_\tau^2 p^{j,k_0} \\
&\quad + \frac{i}{12}\{\partial_\tau^2 q_0^{j,k_0}\cdot\Pi^j_{\{k_0\}} - \partial_\tau^2 sub\ \sigma(P)\}\partial_t\partial_\tau^2 p^{j,k_0} \\
&= P_{m-2}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k_0}\cdot\Pi^j_{\{k_0\}} \\
&\quad + \frac{i}{12}\partial_\tau^2 sub\ \sigma(P)\cdot\partial_t\partial_\tau^2 p^{j,k_0} \\
&\quad + i\sum_{\substack{1\le k<l\le r(j)\\ k,l\ne k_0}} \partial_\tau p^{j,k}\cdot\partial_t p^{j,l}\cdot sub\ \sigma(P^{j,k_0})\Pi^j_{\{k_0,k,l\}} \\
&\quad - \sum_{1\le k\le r(j),\, k\ne k_0} sub\ \sigma(P^{j,k_0}) sub\ \sigma(P^{j,k})\Pi^j_{\{k_0,k\}} \\
&\quad + \frac{i}{2}\sum_{1\le k\le r(j),\, k\ne k_0} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,k_0})\Pi^j_{\{k_0,k\}}
\end{aligned}
$$

$$
\begin{aligned}
&+ i \sum_{k_0<k\le r(j)} \partial_t p^{j,k} \cdot \partial_\tau sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}} \\
&+ i \sum_{1\le k<k_0} \partial_\tau p^{j,k} \cdot \partial_t sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}} \\
&+ \frac{1}{2} \sum_{k_0<k\le r(j)} \partial_t p^{j,k} \cdot \partial_t \partial_\tau^2 p^{j,k_0} \cdot \Pi^j_{\{k_0,k\}} \\
&+ \frac{1}{2} \sum_{1\le k<k_0} \partial_\tau p^{j,k} \cdot \partial_t^2 \partial_\tau p^{j,k_0} \cdot \Pi^j_{\{k_0,k\}} + O(h_{m-2}(t,\tau,\xi)^{1/2}) \\
&+ \frac{i}{12} \{\partial_\tau^2 q_0^{j,k_0} \cdot \Pi^j_{\{k_0\}} - \partial_\tau^2 sub\ \sigma(P)\} \partial_t \partial_\tau^2 p^{j,k_0} \\
= {} & i \sum_{\substack{1\le k<l\le r(j) \\ k,l\ne k_0}} \partial_\tau p^{j,k}(t,\tau,\xi) \cdot \partial_t p^{j,l}(t,\tau,\xi) \cdot sub\ \sigma(P^{j,k_0})(t,\tau,\xi) \\
&\qquad\qquad \times \Pi^j_{\{k_0,k,l\}}(t,\tau,\xi) \\
&- \sum_{1\le k\le r(j),\, k\ne k_0} sub\ \sigma(P^{j,k_0}) sub\ \sigma(P^{j,k}) \Pi^j_{\{k_0,k\}} \\
&+ \frac{i}{2} \sum_{1\le k\le r(j),\, k\ne k_0} \partial_t \partial_\tau p^{j,k} \cdot sub\ \sigma(P^{j,k_0}) \Pi^j_{\{k_0,k\}} \\
&+ i \sum_{1\le k<k_0} \partial_\tau p^{j,k} \cdot \partial_t sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}} \\
&+ i \sum_{k_0<k\le r(j)} \partial_t p^{j,k} \cdot \partial_\tau sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}} \\
&+ \frac{1}{2} \sum_{1\le k<k_0} \partial_\tau p^{j,k} \cdot \partial_t^2 \partial_\tau p^{j,k_0} \cdot \Pi^j_{\{k_0,k\}} + Q(t,\tau,\xi;z^0) \\
&- \frac{1}{12} \sum_{1\le k<k_0} (\partial_t \partial_\tau^2 p^{j,k_0})^2 \partial_\tau p^{j,k} \cdot \Pi^j_{\{k_0,k\}} \\
&- \frac{i}{12} \{2\partial_\tau sub\ \sigma(P^{j,k_0}) \cdot \partial_\tau \Pi^j_{\{k_0\}} + sub\ \sigma(P^{j,k_0}) \partial_\tau^2 \Pi^j_{\{k_0\}}\} \partial_t \partial_\tau^2 p^{j,k_0} \\
&+ O(h_{m-2}(t,\tau,\xi)^{1/2}) \\
&\quad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times I_{j,k_0} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1,
\end{aligned}
$$

since

$$
\begin{aligned}
\partial_\tau \Pi^j_{\{k_0\}}(t,\tau,\xi) &= \sum_{k_0<k\le r(j)} \partial_\tau p^{j,k} \cdot \Pi^j_{\{k_0,k\}} + \sum_{1\le k<k_0} \partial_\tau p^{j,k} \cdot \Pi^j_{\{k_0,k\}}, \\
\partial_t \Pi^j_{\{k_0\}}(t,\tau,\xi) &= \sum_{k_0<k\le r(j)} \partial_t p^{j,k} \cdot \Pi^j_{\{k_0,k\}} + \sum_{1\le k<k_0} \partial_t p^{j,k} \cdot \Pi^j_{\{k_0,k\}}.
\end{aligned}
$$

It follows from (2.14), (2.15), and (2.24) that

$$
\begin{aligned}
(2.25)\quad & sub^2\,\sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) \\
& \equiv Q(t,\tau,\xi;z^0) + \frac{1}{2}\sum_{1\le k<k_0}\{\partial_t^2\partial_\tau p^{j,k_0} - \frac{1}{6}(\partial_t\partial_\tau^2 p^{j,k_0})^2\}\partial_\tau p^{j,k}\cdot \Pi^j_{\{k_0,k\}} \\
& \quad (\bmod\ (\text{L-2}))\ \text{at}\ z^0 \quad \text{for}\ (t,\xi)\in[0,\delta_1]\times\overline{\Gamma}_j\ \text{with}\ |\xi|\ge 1.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&(\partial_t^2\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi) - \frac{1}{6}((\partial_t\partial_\tau^2 p^{j,k_0})(t,-a_1^{j,k_0}/3,\xi))^2 \\
&= \partial_t\{(\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}/3,\xi)\}
\end{aligned}
$$

since

$$
\partial_t a_1^{j,k_0}(t,\xi) = \frac{1}{2}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi).
$$

Modifying $\mathcal{R}(\xi)$ if necessary, we can assume that

$$
\begin{aligned}
&\{\mathrm{Re}\,\lambda;\ \lambda\in\Omega_1 \text{ and } (\partial_t\partial_\tau p^{j,k_0})(\lambda,-a_1^{j,k_0}(\lambda,\xi)/3,\xi)=0\}\subset\mathcal{R}(\xi) \\
&\qquad\qquad \text{if } (\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi)\not\equiv 0 \text{ in } t,
\end{aligned}
$$

where Ω_1 is a compact complex neighborhood of $[0,\delta_1]$. Since, with $C>0$,

$$
\begin{aligned}
&|(\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi)| \le Ch_1(t,-a_1^{j,k_0}/3,\xi;p^{j,k_0})^{1/2}|\xi| \\
&\qquad\qquad \text{for } (t,\xi)\in[0,\delta_1]\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1,
\end{aligned}
$$

Lemma 2.1 and (2.25) give

$$
\begin{aligned}
(2.26)\quad & sub^2\,\sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) \equiv Q(t,\tau,\xi;z^0) \quad (\bmod\ (\text{L-2}))\ \text{at}\ z^0 \\
&\qquad\qquad \text{for } (t,\xi)\in[0,\delta_1]\times\overline{\Gamma}_j \text{ with } |\xi|\ge 1,
\end{aligned}
$$

which proves the assertion (ii). □

2.2. Proof of Theorem 1.2

In this subsection we assume that the conditions (A-1), (A-2), (H) and (T) are satisfied. In order to prove Theorem 1.2 we first derive energy estimates for each factor in (2.4). Fix j with $1\le j\le N_0$, and define

$$
p^{(l)}(t,\tau,\xi) = \partial_\tau^l p(t,\tau,\xi) = \frac{m!}{(m-l)!}\prod_{\mu=1}^{m-l}(\tau-\lambda_\mu^{(l)}(t,\xi)) \quad (\,1\le l\le m-1),
$$

$$p^{j,k(l)}(t,\tau,\xi) = \partial_\tau^l p^{j,k}(t,\tau,\xi) = \frac{m(j,k)!}{(m(j,k)-l)!} \prod_{\mu=1}^{m(j,k)-l} (\tau - \lambda_\mu^{j,k(l)}(t,\xi))$$
$$(1 \le l \le m(j,k)-1)$$

for $(t,\xi) \in [0,\delta_1] \times \overline{\Gamma}_j$ with $|\xi| \ge 1$ and $1 \le k \le r(j)$, where $p(t,\tau,\xi) = \prod_{\mu=1}^m (\tau - \lambda_\mu(t,\xi))$ and $p^{j,k}(t,\tau,\xi) = \prod_{\mu=1}^{m(j,k)} (\tau - \lambda_\mu^{j,k}(t,\xi))$. Here, by Lemma 2.2 we may assume that the $\lambda_\mu(t,\xi)$, $\lambda_\mu^{(l)}(t,\xi)$, $\lambda_\mu^{j,k}(t,\xi)$ and $\lambda_\mu^{j,k(l)}(t,\xi)$ are real analytic in $t \in [0,\delta_1]$. We write, for $1 \le k \le r(j)$,

$$p_l^{j,k}(t,\tau,\xi) = \prod_{1 \le \mu \le m(j,k),\, \mu \ne l} (\tau - \lambda_\mu^{j,k}(t,\xi)) \quad \text{if } m(j,k) = 2 \text{ or } 3,$$
$$p_{i,l}^{j,k}(t,\tau,\xi) = \prod_{1 \le \mu \le m(j,k),\, \mu \ne i,l} (\tau - \lambda_\mu^{j,k}(t,\xi))$$
$$\text{if } i \ne l \text{ and } m(j,k) = 3,$$
$$p_l^{j,k(1)}(t,\tau,\xi) = 3(\tau - \lambda_\mu^{j,k(1)}(t,\xi))$$
$$\text{if } m(j,k) = 3,\ l = 1,2 \text{ and } \{l,\mu\} = \{1,2\},$$

(2.27) $$\mathcal{P}_l^{j,k}(t,\tau,\xi) = p_l^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t \partial_\tau p_l^{j,k}(t,\tau,\xi) \quad \text{if } m(j,k) = 2 \text{ or } 3.$$

Note that $\mathcal{P}_l^{j,k}(t,\tau,\xi) = p_l^{j,k}(t,\tau,\xi)$ if $m(j,k) = 2$.

Lemma **2.6.** (i) (L-1) *for* $[0,\delta_1]$ *is satisfied if and only if there are symbols* $b_{1,l}^{j,k}(t,\xi)$ $(1 \le l \le m(j,k))$ *and* $C > 0$ *such that the* $b_{1,l}^{j,k}(t,\xi)$ *are positively homogeneous of degree* 0 *in* ξ *and*

(2.28) $$sub\, \sigma(P^{j,k})(t,\tau,\xi) = \sum_{l=1}^{m(j,k)} b_{1,l}^{j,k}(t,\xi) p_l^{j,k}(t,\tau,\xi),$$

(2.29) $$\min\{\min_{s \in \mathcal{R}(\xi)} |t-s|, 1\} |b_{1,l}^{j,k}(t,\xi)| \le C \quad (1 \le l \le m(j,k))$$
$$\text{for } (t,\tau,\xi) \in ([0,\delta_1] \setminus \mathcal{R}(\xi)) \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1})$$

provided that $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 2$ *or* 3.
(ii) *Assume that* (L-1) *for* $[0,\delta_1]$ *is satisfied. Then* (L-2) *for* $[0,\delta_1]$ *is satisfied if and only if there are symbols* $b_{2,l}^{j,k}(t,\xi)$ $(l = 1,2)$ *and* $C > 0$ *such that the* $b_{2,l}^{j,k}(t,\xi)$ *are positively homogeneous of degree* 0 *in* ξ *and*

$$sub^2\, \sigma(P^{j,k})(t,\tau,\xi) = \frac{1}{6}(\partial_\tau^2 q_1^{j,k})(t,0,\xi) \sum_{l=1}^{3} p_l^{j,k}(t,\tau,\xi)$$

$$+\sum_{l=1}^{2} b_{2,l}^{j,k}(t,\xi)p_l^{j,k(1)}(t,\tau,\xi),$$

(2.30) $$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|^2,1\}|b_{2,l}^{j,k}(t,\xi)|\le C \quad (\ l=1,2)$$

$$\text{for } (t,\tau,\xi)\in([0,\delta_1]\setminus\mathcal{R}(\xi))\times\mathbf{R}\times(\overline{\Gamma}_j\cap S^{n-1})$$

provided that $1\le j\le N_0$, $1\le k\le r(j)$ *and* $m(j,k)=3$.

PROOF. Since

$$h_{m(j,k)-1}(t,\tau,\xi;p^{j,k})=\sum_{l=1}^{m(j,k)} p_l^{j,k}(t,\tau,\xi)^2$$

$$\text{if } 1\le j\le N_0,\ 1\le k\le r(j) \text{ and } m(j,k)=2 \text{ or } 3,$$

(2.18) and Lemma 2.5 of [11] with $r=m(j,k)$ prove the assertion (i). Let us prove the assertion (ii). Assume that $1\le j\le N_0$, $1\le k\le r(j)$ and $m(j,k)=3$, and put

$$f(t,\tau,\xi)=sub^2\,\sigma(P^{j,k})(t,\tau,\xi)-\frac{1}{2}\partial_\tau^2 q_1^{j,k}(t,\tau,\xi)(\tau^2-(a_1^{j,k}(t,\xi)/3)^2).$$

Note that $\partial_\tau^2 q_1^{j,k}(t,\tau,\xi)$ does not depend on τ. $f(t,\tau,\xi)$ is a polynomial of τ of degree 1 and positively homogeneous of degree 1 in (τ,ξ). Then we can prove that, with some $C_1,C_2>0$,

(2.31) $$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|^2,1\}|f(t,\tau,\xi)|\le C_1h_1(t,\tau,\xi;p^{j,k})^{1/2}$$

$$\text{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\overline{\Gamma}_j\cap S^{n-1})$$

if and only if

$$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|^2,1\}|f(t,-a_1^{j,k}(t,\xi)/3,\xi)|$$
$$\le C_1h_1(t,-a_1^{j,k}(t,\xi)/3,\xi;p^{j,k})^{1/2}\quad \text{for } (t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\cap S^{n-1}).$$

Indeed, we have

$$f(t,\tau,\xi)=f(t,-a_1^{j,k}(t,\xi)/3,\xi)+(\tau+a_1^{j,k}(t,\xi)/3)\partial_\tau f(t,\tau,\xi)$$
$$=f(t,-a_1^{j,k}(t,\xi)/3,\xi)+O(h_1(t,\tau,\xi;p^{j,k})^{1/2}).$$

By (2.4) of [11] we have

$$h_1(t,\tau,\xi;p^{j,k})\le\frac{1}{2}h_1(t,\tau,\xi;p^{j,k(1)})=\frac{1}{2}\sum_{l=1}^{2}p_l^{j,k(1)}(t,\tau,\xi)^2.$$

We have also

$$\tau^2 - (a_1^{j,k}(t,\xi)/3)^2 = \frac{1}{3} p^{j,k(1)}(t,\tau,\xi) - \frac{2}{9} a_1^{j,k}(t,\xi) \sum_{l=1}^{3} (\tau - \lambda_l^{j,k}(t,\xi))$$
$$+ \frac{\sqrt{3}}{18} (a_1^{j,k}(t,\xi)^2/3 - a_2^{j,k}(t,\xi))^{1/2} |p_1^{j,k(1)}(t,\tau,\xi) - p_2^{j,k(1)}(t,\tau,\xi)|,$$
$$\partial_\tau^2 q_1^k(t,\tau,\xi)(\tau^2 - (a_1^{j,k}(t,\xi)/3)^2) = \frac{1}{3} \partial_\tau^2 q_1^k(t,\tau,\xi) p^{j,k(1)}(t,\tau,\xi)$$
$$+ O(h_1(t,\tau,\xi; p^{j,k(1)})^{1/2}).$$

Since

$$f(t, -a_1^{j,k}(t,\xi)/3, \xi) = sub^2\, \sigma(P^{j,k})(t, -a_1^{j,k}(t,\xi)/3, \xi),$$

(2.19), (2.31) and Lemma 2.5 of [11] with $r = 2$ prove the assertion (ii). □

We assume that the hypotheses of Theorem 1.2 are fulfilled. Now let us repeat the same arguments as in §2 and §4 of [11]. Assume that $1 \le j \le N_0$ and $1 \le k \le r(j)$. It is easy to see that

$$(\tau - \lambda_l^{j,k}(t,\xi)) \circ \mathcal{P}_l^{j,k}(t,\tau,\xi) = p^{j,k}(t,\tau,\xi) - \frac{i}{2} \partial_t \partial_\tau p^{j,k}(t,\tau,\xi)$$
$$- \frac{i}{2} \sum_{\mu \ne l} \partial_t (\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi)) \cdot p_{l,\mu}^{j,k}(t,\tau,\xi) - \frac{1}{2} \partial_t^2 \partial_\tau p_l^{j,k}(t,\tau,\xi)$$
$$\text{for } 1 \le l \le 3 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1$$

if $m(j,k) = 3$. So we have

$$(2.32) \qquad (\tau - \lambda_l^{j,k}(t,\xi)) \circ \mathcal{P}_l^{j,k}(t,\tau,\xi)$$
$$= p^{j,k}(t,\tau,\xi) - \frac{i}{2} \partial_t \partial_\tau p^{j,k}(t,\tau,\xi) - \frac{1}{6} \partial_t^2 \partial_\tau^2 p^{j,k}(t,\tau,\xi)$$
$$- \frac{i}{2} \sum_{h \ne l} \partial_t (\lambda_l^{j,k}(t,\xi) - \lambda_h^{j,k}(t,\xi)) \cdot p_{l,h}^{j,k}(t,\tau,\xi)$$
$$- \frac{1}{6} \partial_t^2 \{(\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi)) + (\lambda_l^{j,k}(t,\xi) - \lambda_\nu^{j,k}(t,\xi))\}$$
$$\text{for } 1 \le l \le 3 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1$$

if $m(j,k) = 3$, where $\{l, \mu, \nu\} = \{1,2,3\}$. We have also

$$(\tau - \lambda_l^{j,k}(t,\xi)) \circ p_l^{j,k}(t,\tau,\xi)$$
$$= p^{j,k}(t,\tau,\xi) - \frac{i}{2} \partial_t \partial_\tau p^{j,k}(t,\tau,\xi) - \frac{i}{2} \partial_t (\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi))$$
$$\text{for } l = 1,2 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1$$

if $m(j,k)=2$, where $\{l,\mu\}=\{1,2\}$. Moreover, we have

$$(2.33)\qquad (\tau-\lambda_l^{j,k(1)}(t,\xi))\circ p_l^{j,k(1)}(t,\tau,\xi) = \sum_{\nu=1}^{3}\mathcal{P}_\nu^{j,k}(t,\tau,\xi)-\frac{3i}{2}\partial_t(\lambda_l^{j,k(1)}(t,\xi)-\lambda_\mu^{j,k(1)}(t,\xi))$$

if $m(j,k)=3$, $l=1,2$, $(t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times\overline{\Gamma}_j$, $|\xi|\geq 1$ and $\{l,\mu\}=\{1,2\}$.

(I) Let consider the case where $1\leq k\leq N_0$, $1\leq k\leq r(j)$ and $m(j,k)=3$. Define

$$W_0^{j,k}(t,\xi;\gamma)=\sum_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1+1]}\langle\xi\rangle_\gamma^{3/2}((t-s)^2\langle\xi\rangle_\gamma^{4/3}+1)^{-1/2}, + \sum_{1\leq l<\mu\leq 3}\{(\partial_t(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi)))^2+1\}^{1/2} \times\{(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))^2+1\}^{-1/2}+1,$$

$$W_1^{j,k}(t,\xi) = \sum_{1\leq l<\mu\leq 3}|\partial_t^2(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))|(|\partial_t(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))|+1)^{-1} +|\partial_t(\lambda_2^{j,k(1)}(t,\xi)-\lambda_1^{j,k(1)}(t,\xi))|(|\lambda_2^{j,k(1)}(t,\xi)-\lambda_1^{j,k(1)}(t,\xi)|+1)^{-1},$$

$$\Lambda^{j,k}(t,\xi;\gamma)=\int_0^t(W_0^{j,k}(s,\xi;\gamma)+W_1^{j,k}(s,\xi))\,ds$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^{j,k})$ with $|\xi|\geq 1$ and $\gamma\geq 1$, where $\mathcal{N}^{j,k}=\mathcal{N}_2(p)\cup\mathcal{N}_1(p^{j,k(1)})\cup\{0\}$ and $\langle\xi\rangle_\gamma=(\gamma^2+|\xi|^2)^{1/2}$. It follows from Lemma 2.2, Lemma 2.4 of [11] and Theorem 1 of [9] that there is $C_0>0$ satisfying

$$(2.34)\qquad 0\leq\Lambda^{j,k}(t,\xi;\gamma)\leq C_0(\log\langle\xi\rangle_\gamma+1)$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^{j,k})$ and $\gamma\geq 1$. For $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^{j,k})$ with $|\xi|\geq 1$, $A\geq 1$ and $v(t,\xi)\in C^2([0,\delta_1];L^\infty(\mathbf{R}^n))$ we define

$$\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)=\sum_{l=1}^{3}e^{-A\Lambda^{j,k}}|\mathcal{P}_l^{j,k}v|^2 +\sum_{l=1}^{2}W_0^{j,k}(t,\xi;\gamma)^2e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2+W_0^{j,k}(t,\xi;\gamma)^4e^{-A\Lambda^{j,k}}|v|^2,$$

where $\Lambda^{j,k}=\Lambda^{j,k}(t,\xi;\gamma)$, $\mathcal{P}_l^{j,k}=\mathcal{P}_l^{j,k}(t,D_t,\xi)$ and $p_l^{j,k(1)}=p_l^{j,k(1)}(t,D_t,\xi)$. Then we have

$$(2.35)\qquad D_t\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)$$

$$
\begin{aligned}
&= i\sum_{l=1}^{3}[A\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|\mathcal{P}_l^{j,k}v|^2 + 2\,\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t\mathcal{P}_l^{j,k}v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\}] \\
&\quad + i\sum_{l=1}^{2}[(A(W_0^{j,k})^2\Lambda_t^{j,k} - 2W_0^{j,k}W_{0t}^{j,k})e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2 \\
&\qquad\qquad + 2\,\mathrm{Im}\{(W_0^{j,k})^2e^{-A\Lambda^{j,k}}(D_tp_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\}] \\
&\quad + i[(A(W_0^{j,k})^4\Lambda_t^{j,k} - 4(W_0^{j,k})^3W_{0t}^{j,k})e^{-A\Lambda^{j,k}}|v|^2 \\
&\qquad + 2\,\mathrm{Im}\{(W_0^{j,k})^4e^{-A\Lambda^{j,k}}(D_tv)\cdot\bar{v}\}],
\end{aligned}
$$

where $\Lambda_t^{j,k} = \partial_t\Lambda^{j,k}(t,\xi;\gamma)$ and $W_{0t}^{j,k} = \partial_tW_0^{j,k}(t,\xi;\gamma)$. Since the $\lambda_l^{j,k}(t,\xi)$ and the $\lambda_l^{j,k(1)}(t,\xi)$ are real-valued, it follows from (2.32) and (2.33) that

$$
\begin{aligned}
(2.36)\qquad &\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t\mathcal{P}_l^{j,k}v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\} \\
&= \mathrm{Im}\{e^{-A\Lambda^{j,k}}((D_t-\lambda_l^{j,k})\mathcal{P}_l^{j,k}v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\} \\
&= \mathrm{Im}\{e^{-A\Lambda^{j,k}}((p^{j,k} - \frac{i}{2}(\partial_t\partial_\tau p^{j,k})(t,D_t,\xi))v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\} \\
&\quad - \mathrm{Im}\{e^{-A\Lambda^{j,k}}((\partial_t^2\partial_\tau^2p^{j,k})(t,D_t,\xi)v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\}/6 \\
&\quad - \mathrm{Re}\Big\{e^{-A\Lambda^{j,k}}\sum_{\mu\neq l}(\lambda_{lt}^{j,k} - \lambda_{\mu t}^{j,k})(p_{l,\mu}^{j,k}v)\cdot(\overline{\mathcal{P}_l^{j,k}v})\Big\}/2 \\
&\quad - \mathrm{Im}\Big\{e^{-A\Lambda^{j,k}}\sum_{\mu\neq l}(\lambda_{ltt}^{j,k} - \lambda_{\mu tt}^{j,k})v\cdot(\overline{\mathcal{P}_l^{j,k}v})\Big\}/6, \\
(2.37)\qquad &\mathrm{Im}\{(W_0^{j,k})^2e^{-A\Lambda^{j,k}}(D_tp_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\} \\
&= \mathrm{Im}\{(W_0^{j,k})^2e^{-A\Lambda^{j,k}}((D_t-\lambda_l^{j,k(1)})p_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\} \\
&= \sum_{\mu=1}^{3}\mathrm{Im}\{(W_0^{j,k})^2e^{-A\Lambda^{j,k}}(\mathcal{P}_\mu^{j,k}v)\cdot(\overline{p_l^{j,k(1)}v})\} \\
&\quad - 3\,\mathrm{Re}\{(-1)^l(W_0^{j,k})^2e^{-A\Lambda^{j,k}}(\lambda_{2t}^{j,k(1)} - \lambda_{1t}^{j,k(1)})v\cdot(\overline{p_l^{j,k(1)}v})\}/2, \\
(2.38)\qquad &\mathrm{Im}\{(W_0^{j,k})^4e^{-A\Lambda^{j,k}}(D_tv)\cdot\overline{v}\} \\
&= \mathrm{Im}\Big\{\sum_{l=1}^{2}(W_0^{j,k})^4e^{-A\Lambda^{j,k}}(p_l^{j,k(1)}v)\cdot\overline{v}\Big\}/6,
\end{aligned}
$$

where $\lambda_l^{j,k} = \lambda_l^{j,k}(t,\xi)$, $\lambda_{lt}^{j,k} = \partial_t\lambda_l^{j,k}(t,\xi)$, $\lambda_{ltt}^{j,k} = \partial_t^2\lambda_l^{j,k}(t,\xi)$, $p_{l,\mu}^{j,k} = p_{l,\mu}^{j,k}(t,D_t,\xi)$ and so forth. Put

$$
\hat{f}_\varepsilon(t,\xi) = P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi),
$$

$$q^{j,k}(t,\tau,\xi;\varepsilon)=\sum_{l=0}^{1} q_l^{j,k}(t,\tau,\xi)+r(t,\tau,\xi;\varepsilon)$$

and write

$$\begin{aligned}
&q_1^{j,k}(t,\tau,\xi)+r^{j,k}(t,\tau,\xi;\varepsilon)=\sum_{l=0}^{2}\beta_l^{j,k}(t,\xi;\varepsilon)\tau^{2-l}\\
&\beta_l^{j,k}(t,\xi;\varepsilon)=\beta_{l,0}^{j,k}(t,\xi)+\beta_{l,1}^{j,k}(t,\xi;\varepsilon),
\end{aligned}$$

where $\beta_{l,0}^{j,k}(t,\xi)\in S_{1,0}^{-1+l}([0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\}))$ is positively homogeneous of degree $(-1+l)$ in ξ and $\beta_{l,1}^{j,k}(t,\xi;\varepsilon)\in S_{1,0}^{-2+l}([0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\}))$ uniformly in ε ($l=0,1,2$). Note that

$$q_1^{j,k}(t,\tau,\xi)=\sum_{l=0}^{2}\beta_{l,0}^{j,k}(t,\xi)\tau^{2-l}.$$

Since

$$\begin{aligned}
&|\partial_t W_0^{j,k}(t,\xi;\gamma)|\\
&\le W_0^{j,k}(t,\xi;\gamma)(W_0^{j,k}(t,\xi;\gamma)+\sqrt{2}W_1^{j,k}(t,\xi))\le 2W_0^{j,k}(t,\xi;\gamma)\Lambda_t^{j,k}(t,\xi;\gamma),\\
&p^{j,k}(t,D_t,\xi)v(t,\xi)=P^{j,k}(t,D_t,\xi;\varepsilon)v-q^{j,k}(t,D_t,\xi;\varepsilon)v\\
&=\hat{f}_\varepsilon(t,\xi)-q^{j,k}v,
\end{aligned}$$

(2.35) – (2.38) yield

$$\begin{aligned}
\partial_t\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)&\le 3(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|\hat{f}_\varepsilon(t,\xi)|^2\\
&-\sum_{l=1}^{3}\Big[(A-4)\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|\mathcal{P}_l^{j,k}v|^2\\
&\quad-(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\Big|(q^{j,k}+\frac{i}{2}(\partial_t\partial_\tau p^{j,k})+\frac{1}{6}(\partial_t^2\partial_\tau^2 p^{j,k}))v\Big|^2\\
&\quad-(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\sum_{\mu\ne l}|\lambda_{lt}^{j,k}-\lambda_{\mu t}^{j,k}|^2\{|p_{l,\mu}^{j,k}v|^2/4+(W_1^{j,k})^2|v|^2/36\}\Big]\\
&-(A-10)\sum_{l=1}^{2}(W_0^{j,k})^2\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2\\
&+2(W_0^{j,k})^2(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\Big\{\sum_{l=1}^{3}|\mathcal{P}_l^{j,k}v|^2+\frac{9}{4}(W_1^{j,k})^2|\lambda_2^{j,k(1)}-\lambda_1^{j,k(1)}|^2|v|^2\Big\}
\end{aligned}$$

$$- (A-13)(W_0^{j,k})^4 \Lambda_t^{j,k} e^{-A\Lambda^{j,k}} |v|^2 + \sum_{l=1}^{2} (W_0^{j,k})^4 (\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} |p_l^{j,k(1)} v|^2/6,$$

where $(\partial_t \partial_\tau p^{j,k}) = (\partial_t \partial_\tau p^{j,k})(t, D_t, \xi)$, $\cdots$. First assume that $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$, $|\xi| \geq \gamma \geq 1$ and

$$\min\{ \min_{s \in \mathcal{R}(\xi/|\xi|)} |t-s|, 1\} \leq \langle \xi \rangle_\gamma^{-2/3}. \tag{2.39}$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \geq \langle \xi \rangle_\gamma^{2/3}/\sqrt{2}.$$

So we have, with some $C > 0$,

$$\begin{aligned}
&(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \Big| (q^{j,k} + \frac{i}{2}(\partial_t \partial_\tau p^{j,k}) + \frac{1}{6}(\partial_t^2 \partial_\tau^2 p^{j,k}))v \Big|^2 \\
&\leq C(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \Big\{ \sum_{l=1}^{3} |\mathcal{P}_l^{j,k} v|^2 + \sum_{l=1}^{2} (W_0^{j,k})^3 |p_l^{j,k(1)} v|^2 + (W_0^{j,k})^6 |v|^2 \Big\} \\
&\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \Big\{ \sum_{l=1}^{3} |\mathcal{P}_l^{j,k} v|^2 + \sum_{l=1}^{2} (W_0^{j,k})^2 |p_l^{j,k(1)} v|^2 + (W_0^{j,k})^4 |v|^2 \Big\},
\end{aligned}$$

since there are $c_\mu^{j,k}(t,\xi;\varepsilon) \in S_{1,0}^{\mu}([0,\delta_1] \times (\overline{\Gamma}_j \setminus \{0\}))$ ($\mu = 0,1,2$) uniformly in ε satisfying

$$\begin{aligned}
&q^{j,k}(t,\tau,\xi;\varepsilon) + \frac{i}{2} \partial_t \partial_\tau p^{j,k}(t,\tau,\xi) + \frac{1}{6} \partial_t^2 \partial_\tau^2 p^{j,k}(t,\tau,\xi) \\
&= c_0^{j,k}(t,\xi;\varepsilon) \sum_{l=1}^{3} \mathcal{P}_l^{j,k}(t,\tau,\xi) + c_1^{j,k}(t,\xi;\varepsilon) \sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi) + c_2^{j,k}(t,\xi;\varepsilon).
\end{aligned}$$

Note that there is $C > 0$ such that

$$|\lambda_{lt}^{j,k}(t,\xi)| \leq C|\xi| \quad \text{for } 1 \leq l \leq 3 \text{ and } (t,\xi) \in [0,\delta_1] \times \overline{\Gamma}_j \text{ with } |\xi| \geq 1$$

(see, *e.g.*, Theorem 1 of [9]). Then it follows from (4.11) of [11] that, with $C > 0$,

$$\begin{aligned}
&(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \sum_{\mu \neq l} |\lambda_{lt}^{j,k} - \lambda_{\mu t}^{j,k}|^2 \{ |p_{l,\mu}^{j,k} v|^2/4 + (W_1^{j,k})^2 |v|^2/36 \} \\
&\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \Big\{ \sum_{\mu=1}^{2} (W_0^{j,k})^2 |p_\mu^{j,k(1)} v|^2 + (W_0^{j,k})^4 |v|^2 \Big\} \quad (1 \leq l \leq 3), \\
&2(W_0^{j,k})^2 (\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \Big\{ \sum_{\mu=1}^{3} |\mathcal{P}_\mu^{j,k} v| + 9(W_1^{j,k})^2 |\lambda_2^{j,k(1)} - \lambda_1^{j,k(1)}|^2 |v|^2/4 \Big\}
\end{aligned}$$

$$\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \Big\{ \sum_{\mu=1}^{3} |\mathcal{P}_\mu^{j,k} v|^2 + (W_0^{j,k})^2 \sum_{\mu=1}^{2} |p_\mu^{j,k(1)} v|^2 \Big\}.$$

Therefore, there is $A_0 > 0$ satisfying

$$\partial_t \mathcal{E}^{j,k}(t,\xi;v;\gamma,A) \leq 3|\hat{f}_\varepsilon(t,\xi)|^2 \tag{2.40}$$

for $\varepsilon \in (0,1]$ and $A \geq A_0$ if (2.39) is satisfied. Next assume that $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$, $|\xi| \geq \gamma \geq 1$ and

$$\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|, 1\} \geq \langle\xi\rangle_\gamma^{-2/3}.$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \geq (\sqrt{2}\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1+1]} |t-s|, 1\})^{-1} \tag{2.41}$$

Operating ∂_τ^2 in the both sides of (2.28), we have

$$\partial_\tau^2 sub\ \sigma(P^{j,k})(t,\tau,\xi) = \partial_\tau^2 q_0^{j,k}(t,\tau,\xi) = 2\sum_{l=1}^{3} b_{1,l}^{j,k}(t,\xi). \tag{2.42}$$

Since

$$\begin{aligned}
&\partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi) = -\sum_{\mu\neq l} \lambda_{\mu t}^{j,k}(t,\xi),\\
&\sum_{l=1}^{3} \partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi)\\
&= \partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi) = 2\partial_t a_1^{j,k}(t,\xi) = -2\sum_{l=1}^{3} \lambda_{lt}^{j,k}(t,\xi),\\
&\partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi) - \partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi)/3 = -\sum_{\mu\neq l}(\lambda_{\mu t}^{j,k}(t,\xi) - \lambda_{lt}^{j,k}(t,\xi))/3,\\
&\tau = \sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/6 - a_1^{j,k}(t,\xi)/3,\\
&\sum_{l=1}^{3} \mathcal{P}_l^{j,k}(t,\tau,\xi) = \partial_\tau p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi)\\
&= 3\tau^2 + 2a_1^{j,k}(t,\xi)\tau + a_2^{j,k}(t,\xi) - i\partial_t a_1^{j,k}(t,\xi),\\
&\tau^2 = \sum_{l=1}^{3} \mathcal{P}_l^{j,k}(t,\tau,\xi)/3 - a_1^{j,k}(t,\xi)\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/9
\end{aligned}$$

$$+ 2(a_1^{j,k}(t,\xi)/3)^2 + i\partial_t a_1^{j,k}(t,\xi)/3 - a_2^{j,k}(t,\xi)/3,$$

(2.20), (2.27), (2.42) and Lemma 2.6 give

$$\begin{aligned}
&q^{j,k}(t,\tau,\xi;\varepsilon) + \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi)\\
&= sub\ \sigma(P^{j,k})(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi)\\
&= \sum_{l=1}^{3} b_{1l}^{j,k}(t,\xi)\{\mathcal{P}_l^{j,k}(t,\tau,\xi) + \frac{i}{2}(\partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi) - \frac{1}{3}\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi))\}\\
&\quad + \frac{i}{12}\partial_\tau^2 q_0^{j,k}(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi)\\
&\quad + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi) + \beta_{0,1}^{j,k}(t,\xi;\varepsilon)\tau^2 + \beta_{1,1}^{j,k}(t,\xi;\varepsilon)\tau + \beta_{2,1}^{j,k}(t,\xi;\varepsilon)\\
&= \sum_{l=1}^{3} b_{1,l}^{j,k}(t,\xi)\Big\{\mathcal{P}_l^{j,k}(t,\xi) - \frac{i}{6}\sum_{\mu\neq l}(\lambda_{\mu t}^{j,k}(t,\xi) - \lambda_{lt}^{j,k}(t,\xi))\Big\}\\
&\quad + sub^2\,\sigma(P^{j,k})(t,\tau,\xi) + \beta_{0,1}^{j,k}(t,\xi;\varepsilon)\Big\{\sum_{l=1}^{3}\mathcal{P}_l^{j,k}(t,\tau,\xi)/3\\
&\qquad\qquad - a_1^{j,k}(\cdot)\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/9 + 2(a_1^{j,k}(\cdot)/3)^2 - a_2^{j,k}(\cdot)/3\\
&\qquad\qquad + i\partial_t a_1^{j,k}(t,\xi)/3\Big\}\\
&\quad + \beta_{1,1}(t,\xi;\varepsilon)\Big\{\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/6 - a_1^{j,k}(t,\xi)/3\Big\} + \beta_{2,1}^{j,k}(t,\xi;\varepsilon)\\
&= \sum_{l=1}^{3}(b_{1,l}^{j,k}(t,\xi) + \beta_0^{j,k}(t,\xi;\varepsilon)/3)\mathcal{P}_l^{j,k}(t,\tau,\xi)\\
&\quad - \frac{i}{6}\sum_{l=1}^{3}\sum_{\mu\neq l} b_{1,l}^{j,k}(t,\xi)(\lambda_{\mu t}^{j,k}(t,\xi) - \lambda_{lt}^{j,k}(t,\xi))\\
&\quad + \sum_{l=1}^{2}\{b_{2,l}^{j,k}(t,\xi) + \beta_{1,1}^{j,k}(t,\xi;\varepsilon)/6 - \beta_{0,1}^{j,k}(t,\xi;\varepsilon)a_1^{j,k}(t,\xi)/9\}p_l^{j,k(1)}(t,\xi)\\
&\quad + \gamma_0^{j,k}(t,\xi;\varepsilon),
\end{aligned}$$

where

$$\gamma_0^{j,k}(t,\xi;\varepsilon) = \frac{i}{3}\beta_{0,0}^{j,k}(t,\xi)\partial_t a_1^{j,k}(t,\xi)$$

$$
\begin{aligned}
&+\beta^{j,k}_{0,1}(t,\xi;\varepsilon)(2(a^{j,k}_1(t,\xi)/3)^2-a^{j,k}_2(t,\xi)/3+i\partial_t a^{j,k}_1(t,\xi)/3)\\
&-\beta^{j,k}_{1,1}(t,\xi;\varepsilon)a^{j,k}_1(t,\xi)/3+\beta^{j,k}_{2,1}(t,\xi;\varepsilon)\\
&\in S^0_{1,0}([0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\}))\quad \text{uniformly in } \varepsilon\,(\in(0,1]).
\end{aligned}
$$

By (2.29), (2.30), (2.41) and (4.11) of [11] there is $C>0$ such that

$$
\begin{aligned}
&(\Lambda^{j,k}_t)^{-1}e^{-A\Lambda^{j,k}}|(q^{j,k}+\frac{i}{2}(\partial_t\partial_\tau p^{j,k})+\frac{1}{6}(\partial_t^2\partial_\tau^2 p^{j,k}))v|^2\\
&\le C\Lambda^{j,k}_t e^{-A\Lambda^{j,k}}\Big\{\sum_{l=1}^{3}|\mathcal{P}^{j,k}_l v|^2+\sum_{l=1}^{2}(W^{j,k}_0)^2|p^{j,k(1)}_l v|^2+(W^{j,k}_0)^4|v|^2\Big\}\\
&\qquad \text{for } (t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^{j,k}) \text{ with } |\xi|\ge 1 \text{ and } \varepsilon\in(0,1],
\end{aligned}
$$

since

$$
\begin{aligned}
&|b^{j,k}_{\nu,l}(t,\xi)|\le CW^{j,k}_0(t,\xi;\gamma)^\nu \quad (\nu=1,2),\\
&|(\lambda^{j,k}_{\mu t}(t,\xi)-\lambda^{j,k}_{lt}(t,\xi))v|^2\\
&\le W^{j,k}_0(t,\xi;\gamma)^2(|\lambda^{j,k}_\mu(t,\xi)-\lambda^{j,k}_l(t,\xi)|^2+1)|v|^2\\
&\le W^{j,k}_0(t,\xi;\gamma)^2(|(p^{j,k}_{\mu,\nu}(t,D_t,\xi)-p^{j,k}_{l,\nu}(t,D_t,\xi))v|^2+|v|^2)\\
&\le 2W^{j,k}_0(t,\xi;\gamma)^2\Big(\sum_{h=1}^{2}|p^{j,k(1)}_h v|^2+|v|^2\Big),
\end{aligned}
$$

where $\{l,\mu,\nu\}=\{1,2,3\}$. Similarly, we have, with $C>0$,

$$
\begin{aligned}
&(\Lambda^{j,k}_t)^{-1}e^{-A\Lambda^{j,k}}\sum_{l=1}^{3}\sum_{\mu\ne l}|\lambda^{j,k}_{lt}-\lambda^{j,k}_{\mu t}|^2\{|p^{j,k}_{l,\mu}v|^2/4+(W^{j,k}_1)^2|v|^2/36\}\\
&\le \Lambda^{j,k}_t e^{-A\Lambda^{j,k}}\sum_{l=1}^{3}\sum_{\mu\ne l}(|\lambda^{j,k}_l-\lambda^{j,k}_\mu|^2+1)\{|p^{j,k}_{l,\mu}v|^2/4+(W^{j,k}_0)^2|v|^2/36\}\\
&\le C\Lambda^{j,k}_t e^{-A\Lambda^{j,k}}\Big\{\sum_{l=1}^{3}|\mathcal{P}^{j,k}_l v|^2+(W^{j,k}_0)^2\sum_{l=1}^{2}|p^{j,k(1)}_l v|^2+(W^{j,k}_0)^2|v|^2\Big\},\\
&(W^{j,k}_0)^2(\Lambda^{j,k}_t)^{-1}e^{-A\Lambda^{j,k}}(W^{j,k}_1)^2|\lambda^{j,k(1)}_2-\lambda^{j,k(1)}_2|^2|v|^2\\
&\le 2\Lambda^{j,k}_t e^{-A\Lambda^{j,k}}(W^{j,k}_0)^2\sum_{l=1}^{2}|p^{j,k(1)}_l v|^2/9,
\end{aligned}
$$

since

$$
|(\lambda^{j,k}_l-\lambda^{j,k}_\mu)p^{j,k}_{l,\mu}v|^2=|\mathcal{P}^{j,k}_l v-\mathcal{P}^{j,k}_\mu v+\frac{i}{2}(\lambda^{j,k}_{lt}-\lambda^{j,k}_{\mu t})v|^2
$$

$$\leq 3\Big\{|\mathcal{P}_l^{j,k}v|^2 + |\mathcal{P}_\mu^{j,k}v|^2 + 2(W_0^{j,k})^2\Big(\sum_{h=1}^2 |p_h^{j,k(1)}v|^2/9 + |v|^2\Big)\Big\},$$
$$(\lambda_2^{j,k(1)} - \lambda_1^{j,k(1)})v = (p_2^{j,k(1)} - p_1^{j,k(1)})v/3.$$

Therefore, modifying A_0 if necessary, we can see that (2.40) holds for $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$ with $|\xi| \geq 1$, $\varepsilon \in (0,1]$ and $A \geq A_0$. This gives

$$\mathcal{E}^{j,k}(t,\xi;v;\gamma,A) \leq \mathcal{E}^{j,k}(0,\xi;v;\gamma,A) + 3\int_0^t |\hat{f}_\varepsilon(s,\xi)|^2\,ds \tag{2.43}$$

if $A \geq A_0$, $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$, $|\xi| \geq \gamma \geq 1$ and $\varepsilon \in (0,1]$. We note that A_0 and C_0 in (2.34) depend on $P^{j,k}(t,\tau,\xi;\varepsilon)$.

Lemma **2.7.** *Assume that $1 \leq j \leq N_0$, $1 \leq k \leq r(j)$ and $m(j,k) = 3$. Then there are $c > 0$ and $C_A > 0$ such that*

$$c\mathcal{E}^{j,k}(t,\xi;v;\gamma,A) \leq \sum_{l=0}^2 \langle\xi\rangle_\gamma^{4-2l}|D_t^l v(t,\xi)|^2 \leq C_A\langle\xi\rangle_\gamma^{4+AC_0}\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)$$

for $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$ with $|\xi| \geq \gamma \geq 1$, $\varepsilon \in (0,1]$ and $v \in C^2([0,\delta_1]; L^\infty(\mathbf{R}^n))$.

Proof. We can write

$$D_t^2 v(t,\xi) = \frac{1}{3}\sum_{l=1}^3 \mathcal{P}_l^{j,k}(t,D_t,\xi)v(t,\xi)$$
$$+ \sum_{l=0}^1 c_l(t,\xi)D_t^l v(t,\xi) + \frac{i}{3}\partial_t a_1^{j,k}(t,\xi)\cdot v(t,\xi),$$

where $|c_l(t,\xi)| \leq C|\xi|^{2-l}$. Similarly, we have

$$D_t v(t,\xi) = \frac{1}{6}\sum_{l=1}^2 p_l^{j,k(1)}(t,D_t,\xi)v(t,\xi) + d^{j,k}(t,\xi)v(t,\xi),$$

where $|d^{j,k}(t,\xi)| \leq C|\xi|$. Therefore, we have

$$\sum_{l=0}^2 \langle\xi\rangle_\gamma^{4-2l}|D_t^l v(t,\xi)|^2$$
$$\leq C_A\langle\xi\rangle_\gamma^{4+AC_0}e^{-A\Lambda^{j,k}}\Big\{\sum_{l=1}^3 |\mathcal{P}_l^{j,k}v|^2 + \sum_{l=1}^2 (W_0^{j,k})^2|p_l^{j,k(1)}v|^2 + (W_0^{j,k})^4|v|^2\Big\}$$

$$\leq C_A \langle\xi\rangle_\gamma^{4+AC_0} \mathcal{E}^{j,k}(t,\xi;v;\gamma,A).$$

It is obvious that, with $C > 0$,

$$\mathcal{E}^{j,k}(t,\xi;v;\gamma,A) \leq C \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{4-2l} |D_t^l v(t,\xi)|^2,$$

since $W_0^{j,k}(t,\xi;\gamma) \leq C\langle\xi\rangle_\gamma^{2/3}$. □

Lemma **2.8.** *Assume that $1 \leq j \leq N_0$, $1 \leq k \leq r(j)$ and $m(j,k) = 3$. Then for $\mu \in \mathbf{N}$ with $\mu \geq 2$ and $\kappa \in \mathbf{R}$ there are $\nu_{j,k} > 0$ and $C_\mu > 0$ such that*

$$\begin{aligned} &\sum_{l=0}^{\mu} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_t^l v(t,\xi)|^2 \\ &\leq C_\mu \Big\{ \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{2\mu+2\kappa+4+\nu_{j,k}-2l} |(D_t^l v)(0,\xi)|^2 \\ &\qquad + \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa+\nu_{j,k}} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds \\ &\qquad + \sum_{l=0}^{\mu-3} \langle\xi\rangle_\gamma^{2\mu+2\kappa-6-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \Big\} \end{aligned} \tag{2.44}$$

for $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma}_j \setminus \mathcal{N}^{j,k})$ with $|\xi| \geq \gamma \geq 1$, $\varepsilon \in (0,1]$ and $v \in C^\infty([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where $\sum_{l=0}^{\mu-3} \cdots = 0$ when $\mu = 2$ and the $\nu_{j,k}$ do not depend on μ.

Proof. From (2.43) with $A = A_0$ and Lemma 2.7 with $A = A_0$ it follows that (2.44) is valid for $\mu = 2$ if $\nu_{j,k} \geq A_0 C_0$. Let $M \geq 2$, and assume that (2.44) is valid for $\mu = M$. Then we have

$$\begin{aligned} &\sum_{l=0}^{M+1} \langle\xi\rangle_\gamma^{2M+2+2\kappa-2l} |D_t^l v(t,\xi)|^2 \\ &\leq C_M \Big\{ \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{2M+2+2\kappa+4+\nu_{j,k}-2l} |(D_t^l v)(0,\xi)|^2 \\ &\qquad + \int_0^t \langle\xi\rangle_\gamma^{2M+2+2\kappa+\nu_{j,k}} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds \\ &\qquad + \sum_{l=0}^{\mu-3} \langle\xi\rangle_\gamma^{2M+2+2\kappa-6-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \Big\} \end{aligned} \tag{2.45}$$

$$+\langle\xi\rangle_\gamma^{2\kappa}|D_t^{M+1}v(t,\xi)|^2$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^{j,k})$ with $|\xi|\ge\gamma\ge 1$, $\varepsilon\in(0,1]$ and $v\in C^\infty([0,\delta_1];L^\infty(\mathbf{R}^n))$. On the other hand, we have

$$D_t^3v(t,\xi)=-\sum_{l=0}^2 a_{3-l}^{j,k}(t,\xi;\varepsilon)D_t^lv(t,\xi)+P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi),$$

where $P^{j,k}(t,\tau,\xi;\varepsilon)=\tau^3+\sum_{l=1}^3 a_l^{j,k}(t,\xi;\varepsilon)\tau^{3-l}$. By induction we can easily show that for $h\in\mathbf{Z}_+$ there are symbols $a_{3+h-l}^{j,k,h}(t,\xi;\varepsilon)\in S_{1,0}^{3+h-l}([0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\}))$ ($l=0,1,2$) and $b_{h-l}^{j,k,h}(t,\xi;\varepsilon)\in S_{1,0}^{h-l}([0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\}))$ uniformly in $\varepsilon\in(0,1]$ ($0\le l\le h$) satisfying

$$\begin{aligned}D_t^{3+h}v(t,\xi)=&\sum_{l=0}^2 a_{3+h-l}^{j,k,h}(t,\xi;\varepsilon)D_t^lv(t,\xi)\\&+\sum_{l=0}^h b_{h-l}^{j,k,h}(t,\xi;\varepsilon)D_t^lP^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi).\end{aligned}$$

This, with (2.44) for $\mu=2$ and (2.45), proves that (2.44) is valid for $\mu=M+1$. □

(II) Next consider the case where $1\le j\le N_0$, $1\le k\le r(j)$ and $m(j,k)=2$. Define

$$\begin{aligned}&W_0^{j,k}(t,\xi;\gamma)=\sum_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1]}\langle\xi\rangle_\gamma^{1/2}((t-s)^2\langle\xi\rangle_\gamma+1)^{-1/2}+1,\\&W_1^{j,k}(t,\xi)=|\partial_t(\lambda_1^{j,k}(t,\xi)-\lambda_2^{j,k}(t,\xi))|/(|\lambda_1^{j,k}(t,\xi)-\lambda_2^{j,k}(t,\xi)|+1),\\&\Lambda^{j,k}(t,\xi;\gamma)=\int_0^t(W_0(s,\xi;\gamma)+W_1(s,\xi))\,ds\end{aligned}$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0)$ with $|\xi|\ge 1$ and $\gamma\ge 1$, where $\mathcal{N}^0=\mathcal{N}_2(p)\cup\{0\}$. Similarly, we have

$$\begin{aligned}&|\partial_tW_0^{j,k}(t,\xi;\gamma)|\le W_0^{j,k}(t,\xi;\gamma)^2,\\&0\le\Lambda^{j,k}(t,\xi;\gamma)\le C_0(\log\langle\xi\rangle_\gamma+1)\end{aligned}$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0)$ with $|\xi|\ge 1$, where $C_0>0$. For $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0)$ with $|\xi|\ge 1$, $A\ge 1$ and $v(t,\xi)\in C^1([0,\delta_1];L^\infty(\mathbf{R}^n))$ we define

$$\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)=\sum_{l=1}^2 e^{-A\Lambda^{j,k}}|p_l^{j,k}v|^2+W_0^{j,k}(t,\xi;\gamma)^2e^{-A\Lambda^{j,k}}|v|^2,$$

where $\Lambda^{j,k} = \Lambda^{j,k}(t,\xi;\gamma)$ and $p_l^{j,k} = p_l^{j,k}(t, D_t, \xi)$. Then we have

$$
\begin{aligned}
&D_t\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)\\
&= i\sum_{l=1}^{2} A\Lambda_t^{j,k} e^{-A\Lambda^{j,k}}|p_l^{j,k}v|^2 + 2i\,\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t p_l^{j,k}v)\cdot(\overline{p_l^{j,k}v})\}\\
&\quad + i(A\Lambda_t^{j,k}(W_0^{j,k})^2 - 2W_0^{j,k}W_{0t}^{j,k})e^{-A\Lambda^{j,k}}|v|^2\\
&\quad + 2i\,\mathrm{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}(D_t v)\cdot\bar{v}\},
\end{aligned}
$$

where $\Lambda_t^{j,k} = \partial_t\Lambda^{j,k}(t,\xi;\gamma)$, $W_0^{j,k} = W_0^{j,k}(t,\xi;\gamma)$ and $W_{0t}^{j,k} = \partial_t W_0^{j,k}(t,\xi;\gamma)$. Put

$$
\begin{aligned}
&\hat{f}_\varepsilon(t,\xi) = P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)\\
&P^{j,k}(t,\tau,\xi;\varepsilon) = p^{j,k}(t,\tau,\xi) + q^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon),
\end{aligned}
$$

where $q^{j,k}(t,\tau,\xi) \in \mathcal{S}^1_{1,0}([0,\delta_1]\times(\overline{\Gamma}_j\backslash\mathcal{N}^0))$ is positively homogeneous of degree 1 for $|\xi| \geq 1$ and $r^{j,k}(t,\tau,\xi;\varepsilon) \in \mathcal{S}^{1,-1}_{1,0}([0,\delta_1]\times(\overline{\Gamma}_j\backslash\mathcal{N}^0))$ uniformly in ε. Then we have

$$
\begin{aligned}
(2.46)\quad &\partial_t\mathcal{E}^{j,k}(t,\xi;v;\gamma,A)\\
&\leq -\sum_{l=1}^{2}[A\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|p_l^{j,k}v|^2\\
&\qquad\qquad - 2\,\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t - \lambda_l^{j,k})p_l^{j,k}v\cdot(\overline{p_l^{j,k}v})\}]\\
&\quad - \{A\Lambda_t^{j,k}(W_0^{j,k})^2 - 2(W_0^{j,k})^3\}e^{-A\Lambda^{j,k}}|v|^2\\
&\quad - \mathrm{Im}\Big\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}\sum_{l}^{2} p_l^{j,k}v\cdot\bar{v}\Big\}\\
&\leq -\sum_{l=1}^{2} e^{-A\Lambda^{j,k}}[A\Lambda_t^{j,k}|p_l^{j,k}v|^2 - (\Lambda_t^{j,k})^{-1}|\hat{f}_\varepsilon|^2 - 3\Lambda_t^{j,k}|p_l^{j,k}v|^2\\
&\qquad\qquad - 3(\Lambda_t^{j,k})^{-1}|(sub\ \sigma(P^{j,k})(t,D_t,\xi)\\
&\qquad\qquad\qquad + (-1)^l\frac{i}{2}\partial_t(\lambda_2^{j,k} - \lambda_1^{j,k}))v|^2\\
&\qquad\qquad - 3(\Lambda_t^{j,k})^{-1}|r^{j,k}v|^2]\\
&\quad - \{A\Lambda_t^{j,k}(W_0^{j,k})^2 - 2(W_0^{j,k})^3\}e^{-A\Lambda^{j,k}}|v|^2\\
&\quad + (\Lambda_t^{j,k})^{-1}(W_0^{j,k})^2e^{-A\Lambda^{j,k}}\sum_{l=1}^{2}|p_l^{j,k}v|^2 + \frac{1}{2}\Lambda_t^{j,k}(W_0^{j,k})^2e^{-A\Lambda^{j,k}}|v|^2
\end{aligned}
$$

$$\begin{aligned}
&\le 2(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|\hat{f}_\varepsilon|^2 - \sum_{l=1}^{2}(A-4)\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|p_l^{j,k}v|^2\\
&\quad + 12(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|sub\ \sigma(P^{j,k})v|^2\\
&\quad + 3(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|(\lambda_{1t}^{j,k}-\lambda_{2t}^{j,k})v|^2 + 6(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|r^{j,k}v|^2\\
&\quad - (A-5/2)\Lambda_t^{j,k}(W_0^{j,k})^2e^{-A\Lambda^{j,k}}|v|^2
\end{aligned}$$

since

$$\begin{aligned}
&(\tau-\lambda_l^{j,k}(t,\xi))\circ p_l^{j,k}(t,\tau,\xi)\\
&= P^{j,k}(t,\tau,\xi;\varepsilon) - q^{j,k}(t,\tau,\xi) - r^{j,k}(t,\tau,\xi;\varepsilon) - i\partial_t p_l^{j,k}(t,\tau,\xi),\\
&- i\partial_t p_l^{j,k}(t,\tau,\xi) = (-1)^l\frac{i}{2}\partial_t(\lambda_1^{j,k}(t,\xi)-\lambda_2^{j,k}(t,\xi)) - \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi)\\
&\quad (\ l=1,2),
\end{aligned}$$

where $p_l^{j,k} = p_l^{j,k}(t,D_t,\xi)$, $\lambda_l^{j,k} = \lambda_l^{j,k}(t,\xi)$, $sub\ \sigma(P^{j,k}) = sub\ \sigma(P^{j,k})(t,D_t,\xi)$, $\lambda_{lt}^{j,k} = \partial_t\lambda_l^{j,k}(t,\xi)$ and so forth. It is easy to see that

$$|(\lambda_{1t}^{j,k}(t,\xi)-\lambda_{2t}^{j,k}(t,\xi))v(t,\xi)|^2 \le 4(\Lambda_t^{j,k})^2\sum_{l=1}^{2}|p_l^{j,k}v|^2 + 2(\Lambda_t^{j,k})^2|v|^2, \tag{2.47}$$

$$|r^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \le C\Big\{|\xi|^{-2}\sum_{l=1}^{2}|p_l^{j,k}v|^2 + |v|^2\Big\} \tag{2.48}$$

$$\text{for } (t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0) \text{ with } |\xi|\ge 1 \text{ and } \varepsilon\in(0,1],$$

where $C>0$. First assume that

$$\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1+1]}|t-s|,1\} \le \langle\xi\rangle_\gamma^{-1/2}. \tag{2.49}$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \ge \langle\xi\rangle_\gamma^{1/2}/\sqrt{2}.$$

Therefore, there is $A_0>0$ satisfying

$$\partial_t\mathcal{E}^{j,k}(t,\xi;v;\gamma,A) \le 2|\hat{f}_\varepsilon(t,\xi)|^2 \tag{2.50}$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0)$ with $|\xi|\ge 1$, $\varepsilon\in(0,1]$ and $A\ge A_0$ if (2.49) is satisfied. Next assume that

$$\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1+1]}|t-s|,1\} \ge \langle\xi\rangle_\gamma^{-1/2}.$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \ge (\sqrt{2}\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)\cap[0,\delta_1+1]}|t-s|,1\})^{-1}.$$

Lemma 2.6 (i) and (2.46) – (2.48) prove that (2.50) is valid for $(t,\xi) \in [0,\delta_1]\times(\overline{\Gamma}_j \setminus \mathcal{N}^0)$ with $|\xi|\ge 1$, $\varepsilon\in(0,1]$ and $A \ge A_0$, with a modification of A_0 if necessary. Repeating the same argument as in Lemma 2.8, we have the following

Lemma **2.9.** *Assume that $1\le j\le N_0$, $1\le k\le r(j)$ and $m(j,k)=2$. Then for $\mu\in\mathbf{N}$ and $\kappa\in\mathbf{R}$ there are $\nu_{j,k}>0$ and $C_\mu>0$ such that*

$$\begin{aligned}
&\sum_{l=0}^{\mu}\langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_t^l v(t,\xi)|^2\\
&\le C_\mu\Big\{\sum_{l=0}^{1}\langle\xi\rangle_\gamma^{2\mu+2\kappa+2+\nu_{j,k}-2l}|(D_t^l v)(0,\xi)|^2\\
&\quad+\int_0^t\langle\xi\rangle_\gamma^{2\mu+2\kappa+\nu_{j,k}}|P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds\\
&\quad+\sum_{l=0}^{\mu-2}\langle\xi\rangle_\gamma^{2\mu+2\kappa-4-2l}|D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2\Big\}
\end{aligned}$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\setminus\mathcal{N}^0)$ with $|\xi|\ge\gamma\ge 1$, $\varepsilon\in(0,1]$ and $v\in C^\infty([0,\delta_1];L^\infty(\mathbf{R}^n))$, where $\sum_{l=0}^{\mu-2}\cdots=0$ when $\mu=1$ and the $\nu_{j,k}$ do not depend on μ.

(III) Now consider the case where $1\le j\le N_0$, $1\le k\le r(j)$ and $m(j,k)=1$. Define

$$\mathcal{E}^{j,k}(t,\xi;v;A)=e^{-At}|v(t,\xi)|^2$$

for $(t,\xi)\in[0,\delta_1]\times\overline{\Gamma}_j$ with $|\xi|\ge 1$, $A\ge 1$ and $v(t,\xi)\in C([0,\delta_1];L^\infty(\mathbf{R}^n))$. Then we have

$$D_t\mathcal{E}^{j,k}(t,\xi;v;A)=iAe^{-At}|v(t,\xi)|^2+2i\,\mathrm{Im}\{e^{-At}p^{j,k}v\cdot\bar{v}\},$$

where $p^{j,k}=p^{j,k}(t,D_t,\xi)\,(=D_t-\lambda^{j,k}(t,\xi))$. Applying the same argument as in the proof of Lemma 2.8, we can prove the following

Lemma **2.10.** *Assume that $1\le j\le N_0$, $1\le k\le r(j)$ and $m(j,k)=1$. Then for $\mu\in\mathbf{Z}_+$ and $\kappa\in\mathbf{R}$ there is $C_\mu>0$ such that*

$$\sum_{l=0}^{\mu}\langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_t^l v(t,\xi)|^2\le C_\mu\{\langle\xi\rangle_\gamma^{2\mu+2\kappa}|v(0,\xi)|^2$$

$$+\int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa}|P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$
$$+\sum_{l=0}^{\mu-1}\langle\xi\rangle_\gamma^{2\mu+2\kappa-2-2l}|D_t^lP^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2\}$$

for $(t,\xi)\in[0,\delta_1]\times\overline{\Gamma}_j$ *with* $|\xi|\geq\gamma\geq 1$, $\varepsilon\in(0,1]$ *and* $v\in C^\infty([0,\delta_1];L^\infty(\mathbf{R}^n))$, *where* $\sum_{l=0}^{\mu-1}\cdots=0$ *when* $\mu=0$.

Let $f(t,x)\in C^\infty(\mathbf{R};\mathcal{S}(\mathbf{R}^n_x))$ satisfy $\operatorname{supp} f\subset\{(t,x)\in\mathbf{R}\times\mathbf{R}^n;\ t\geq 0\}$, and consider the Cauchy problem

$$\text{(CP)}'\qquad \begin{cases} P(t,D_t,D_x)u(t,x)=f(t,x),\\ u(t,x)|_{t<0}=0.\end{cases}$$

Put

$$\widetilde{\Gamma}_1=\Gamma_1,\qquad \widetilde{\Gamma}_j=\Gamma_j\setminus\bigcup_{l=1}^{j-1}\Gamma_l\quad(\ 2\leq j\leq N_0),$$
$$\widetilde{\mathcal{N}}=\bigcup_{j=1}^{N_0}(\bigcup_{1\leq k\leq r(j),\,m(j,k)=3}\mathcal{N}^{j,k})\cup\mathcal{N}_2(p)\cup\{0\}.$$

Let $v_0(t,\xi;\varepsilon)\in C^\infty(\mathbf{R};\mathcal{S}(\mathbf{R}^n_\xi))$ satisfy $v_0(t,\xi;\varepsilon)|_{t<0}=0$, and define

$$v_{k+1}(t,\xi;\varepsilon)=P^{j,r(j)-k}(t,D_t,\xi;\varepsilon)v_k(t,\xi;\varepsilon)$$
$$\text{for } 1\leq j\leq N_0,\ \xi\in\widetilde{\Gamma}_j\setminus\widetilde{\mathcal{N}} \text{ and } 0\leq k\leq r(j)-1.$$

Then it follows from Lemmas 2.8 – 2.10 that for $1\leq j\leq N_0$, $0\leq k\leq r(j)-1$, $\mu\geq m(j,r(j)-k)$, $\kappa\in\mathbf{R}$, $\gamma\geq 1$ and $(t,\xi)\in[0,\delta_1]\times(\widetilde{\Gamma}_j\setminus\widetilde{\mathcal{N}})$

$$\sum_{l=0}^{\mu}\int_0^t\langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_s^lv_k(s,\xi;\varepsilon)|^2\,ds$$
$$\leq C_\mu\sum_{l=0}^{\mu-m(j,r(j)-k)}\int_0^t\langle\xi\rangle_\gamma^{2\mu+2\kappa+\tilde{\nu}_{j,k}-2l}|D_s^lv_{k+1}(s,\xi;\varepsilon)|^2\,ds,$$

where $C_\mu>0$, $\tilde{\nu}_{j,k}=0$ if $m(j,r(j)-k)=1$ and $\tilde{\nu}_{j,k}=\nu_{j,r(j)-k}$ if $m(j,r(j)-k)=2$ or 3. This yields

$$\sum_{l=0}^{\mu}\int_0^t\langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_s^lv_0(s,\xi;\varepsilon)|^2\,ds$$

$$\leq C_\mu \sum_{l=0}^{\mu-m} \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l+\tilde{\nu}} |D_s^l v_{r(j)}(s,\xi;\varepsilon)|^2\, ds$$

for $1 \leq j \leq N_0$, $\mu \geq m$, $\kappa \in \mathbf{R}$, $\gamma \geq 1$ and $(t,\xi) \in [0,\delta_1] \times (\widetilde{\Gamma}_j \setminus \widetilde{\mathcal{N}})$, where $C_\mu > 0$ and $\tilde{\nu} = \max_{1\leq j\leq N_0}(\tilde{\nu}_{j,0} + \tilde{\nu}_{j,1} + \cdots + \tilde{\nu}_{j,r(j)-1})$. By (2.4) we can see that there are $C > 0$ and $C_N > 0$ ($N = 0,1,2,\cdots$) satisfying

$$\sum_{l=0}^{m} \int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l} |D_s^l v(s,\xi;\varepsilon)|^2\, ds$$
$$\leq C \int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa+\tilde{\nu}} |P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds$$
$$+ C_N \sum_{l=0}^{m-1} \int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l-N} |D_s^l v(s,\xi)|^2\, ds$$

for $\kappa \in \mathbf{R}$, $\gamma \geq 1$, $(t,\xi) \in [0,\delta_1] \times (\mathbf{R}^n \setminus \widetilde{\mathcal{N}})$ and $N = 0,1,2,\cdots$. Therefore, taking $\gamma_0 = 2C_1$ and modifying $\tilde{\nu}$ if necessary, we have

$$\sum_{l=0}^{m} \int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l} |D_s^l v(s,\xi)|^2\, ds$$
$$\leq 2C \int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa+\tilde{\nu}} |P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds$$

for $v(t,\xi) \in C^\infty(\mathbf{R};\mathcal{S}(\mathbf{R}^n_\xi))$ with $v(t,\xi)|_{t<0} = 0$ if $\kappa \in \mathbf{R}$, $(t,\xi) \in [0,\delta_1] \times (\mathbf{R}^n \setminus \widetilde{\mathcal{N}})$, $\varepsilon \in (0,1]$ and $|\xi| \geq \gamma \geq \gamma_0$. Noting that $P(t,\tau,\xi;\varepsilon) - \tau^m \in \mathcal{S}^{m-1,1}_{1,0}(\mathbf{R} \times \mathbf{R}^n)$ uniformly in ε, similarly we can prove the following

LEMMA **2.11.** *There are $C_\mu > 0$ ($\mu \geq m$) such that*

$$\sum_{l=0}^{\mu} \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_s^l v(s,\xi)|^2\, ds$$
$$\leq C_\mu \sum_{l=0}^{\mu-m} \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l+\tilde{\nu}} |D_s^l P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds$$

for $\mu \geq m$, $\kappa \in \mathbf{R}$, $v(t,\xi) \in C^\infty(\mathbf{R};\mathcal{S}(\mathbf{R}^n_\xi))$ with $v(t,\xi)|_{t<0} = 0$, $(t,\xi) \in [0,\delta_1] \times (\mathbf{R}^n \setminus \widetilde{\mathcal{N}})$ with $|\xi| \geq \gamma \geq \gamma_0$ and $\varepsilon \in (0,1]$.

(IV) Let us derive energy estimates for $|\xi| \leq \gamma$. Define

$$\mathcal{E}^0(t,\xi;v;\gamma,A) = \sum_{l=0}^{m-1} e^{-At} \langle\xi\rangle_\gamma^{2m-2-2l} |D_t^l v(t,\xi)|^2$$

for $(t,\xi) \in [0,\delta_1] \times \mathbf{R}^n$ with $|\xi| \le \gamma$ and $v(t,\xi) \in C^m([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where $A \ge 1$ and $\gamma \ge \gamma_0$. Then we have

$$\begin{aligned} D_t\mathcal{E}^0(t,\xi;v;\gamma,A) &= \sum_{l=0}^{m-1} iAe^{-At}\langle\xi\rangle_\gamma^{2m-2-2l}|D_t^l v(t,\xi)|^2 \\ &\quad + 2ie^{-At}\,\mathrm{Im}\{D_t^m v\cdot\overline{(D_t^{m-1}v)}\} + \sum_{l=0}^{m-2} 2i\langle\xi\rangle_\gamma^{2m-2-2l}e^{-At}\,\mathrm{Im}\{D_t^{l+1}v\cdot\overline{(D_t^l v)}\}. \end{aligned}$$

Since $P(t,\tau,\xi;\varepsilon)-\tau^m \in \mathcal{S}_{1,0}^{m-1,1}(\mathbf{R}\times\mathbf{R}^n)$ uniformly in ε, there is $C_0>0$ such that

$$\partial_t\mathcal{E}^0(t,\xi;v;\gamma,A) \le 4A^{-1}e^{-At}|P(t,D_t,\xi;\varepsilon)v(t,\xi)|^2$$

if $A \ge C_0\gamma$ and $|\xi|\le\gamma$. This yields

$$\begin{aligned} &\sum_{l=0}^{m-1}\langle\xi\rangle_\gamma^{2m+2\kappa-2-2l}|D_t^l v(t,\xi)|^2 \\ &\le C_\gamma\Big\{\sum_{l=0}^{m-1}\langle\xi\rangle_\gamma^{2m+2\kappa-2-2l}|(D_t^l v)(0,\xi)|^2 + \int_0^t \langle\xi\rangle_\gamma^{2\kappa}|P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds\Big\} \end{aligned}$$

for $(t,\xi) \in [0,\delta_1]\times\mathbf{R}^n$ with $|\xi|\le\gamma$, $\varepsilon\in(0,1]$ and $v(t,\xi) \in C^m([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where C_γ is a positive constant depending on γ. Similarly, for $\mu \ge m-1$ there are $C_{\gamma,\mu}>0$ ($\mu\ge m-1$) such that

$$\begin{aligned} &\sum_{l=0}^{\mu}\langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_t^l v(t,\xi)|^2 \\ &\le C_{\gamma,\mu}\Big\{\sum_{l=0}^{m-1}\langle\xi\rangle_\gamma^{2m+2\kappa-2l}|(D_t^l v)(0,\xi)|^2 \\ &\qquad + \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2m+2}|P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds \\ &\qquad + \sum_{l=0}^{\mu-m}\langle\xi\rangle_\gamma^{2\mu+2\kappa-2m-2l}|D_t^l P(t,D_t,\xi;\varepsilon)v(t,\xi)|^2\Big\} \end{aligned}$$

for $(t,\xi) \in [0,\delta_1]\times\mathbf{R}^n$ with $|\xi|\le\gamma$, $\varepsilon\in(0,1]$ and $v(t,\xi) \in C^m([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where $\sum_{l=0}^{-1}\cdots = 0$. This, together with Lemmas 2.8 – 2.11, yields the following

LEMMA **2.12.** *There are $\gamma_0 \ge 1$, $C_{\gamma,\mu}>0$ ($\gamma\ge\gamma_0$, $\mu\ge m$) and $\nu_0>0$ such that*

$$\sum_{l=0}^{\mu}\|\langle D_x\rangle_\gamma^{\mu+\kappa-l}D_t^l u(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)} \tag{2.51}$$

$$\leq C_{\gamma,\mu} \sum_{l=0}^{\mu-m} \|\langle D_x\rangle_\gamma^{\mu+\kappa-m-l+\nu_0} D_t^l P(t,D_t,D_x;\varepsilon)u(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}$$

if $\mu \geq m$, $\gamma \geq \gamma_0$, $\varepsilon \in (0,1]$ *and* $u(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n))$ *with* $u(t,x)|_{t<0} = 0$*. Here* $H^s(\mathbf{R}^n)$ *denotes the Sobolev space of order* s *and* $H^\infty(\mathbf{R}^n) = \cap_{s\in\mathbf{R}} H^s(\mathbf{R}^n)$ *and*

$$\|f(t,x)\|_{L^2([0,\delta_1]\times\mathbf{R}^n)} = \Big(\int_{[0,\delta_1]\times\mathbf{R}^n} |f(t,x)|^2\, dtdx\Big)^{1/2}.$$

Remark. (2.51) is valid, replacing $P(t,D_t,D_x;\varepsilon)$ by $P(t,D_t,D_x)$.

Let $f(t,x) \in C^\infty([0,\infty); H^\infty(\mathbf{R}^n))$ satisfy $(D_t^j f)(0,x) = 0$ for $j \in \mathbf{Z}_+$. Then, it follows from the unique existence theorem for ordinary differential equations and the proof of Lemma 2.12 with $P(t,D_t,D_x;\varepsilon)$ replaced by $P(t,D_t,D_x)$ that the Cauchy problem

$$(\mathrm{CP})_0 \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\delta_1]\times\mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (\ 0 \leq j \leq m-1) \end{cases}$$

has a unique solution $u(t,x) \in C^\infty([0,\delta_1]; H^\infty(\mathbf{R}^n))$. We note that $(\mathrm{CP})_0$ has a unique solution $u(t,x) \in C^\infty([0,\delta_1]; H^\infty(\mathbf{R}^n))$ even if $P(t,D_t,D_x)$ is replaced by $P(t,D_t,D_x;\varepsilon)$.

Lemma **2.13.** *Let* $u \in C^\infty((-\infty,\delta_1]\times\mathbf{R}^n)$ *satisfy* $u(t,x)|_{t<0} = 0$*, and let* $(t_0,x^0) \in [0,\delta_1]\times\mathbf{R}^n$*. Then* $(t_0,x^0) \notin \operatorname{supp} u$ *if*

$$K^-_{(t_0,x^0)} \cap \operatorname{supp} P(t,D_t,D_x)u = \emptyset. \tag{2.52}$$

Proof. We extend $u(t,x)$ to a function in $C^\infty(\mathbf{R}^{n+1})$. Choose $R > 0$ so that

$$K^-_{(t_0,x^0)} \subset \{(t,x) \in [0,\delta_1]\times\mathbf{R}^n;\ |x| \leq R\}.$$

Assume that (2.52) is valid. Let $\Theta(t)$ be a function in $\mathcal{E}^{\{\kappa_0\}}(\mathbf{R})$ satisfying

$$\Theta(t) = \begin{cases} 1 & \text{if } t \leq 3/2, \\ 0 & \text{if } t \geq 2. \end{cases}$$

Put

$$F_R(t,x) = \Theta(|x|-R)P(t,D_t,D_x)u(t,x) + [P, \Theta(|x|-R)]u(t,x),$$

where $[A,B] = AB - BA$. Then we have

$$P(t,D_t,D_x)(\Theta(|x|-R)u(t,x)) = F_R(t,x).$$

Note that $F_R(t,x)|_{t<0} = 0$. It is easy to see that there is a unique solution $v_R(t,x) \in C^\infty((-\infty,\delta_1]; H^\infty(\mathbf{R}^n)$ satisfying

$$(\mathrm{CP})_R \qquad \begin{cases} P(t,D_t,D_x)v_R(t,x) = F_R(t,x) & \text{in } (-\infty,\delta_1] \times \mathbf{R}^n, \\ v_R(t,x)|_{t<0} = 0. \end{cases}$$

Therefore, we have $v_R(t,x) = \Theta(|x| - R)u(t,x)$ for $t \in (-\infty,\delta_1]$. Choose $\rho^1(t) \in \mathcal{E}^{\{\kappa_0\}}(\mathbf{R})$ and $\rho^n(x) \in \mathcal{E}^{\{\kappa_0\}}(\mathbf{R}^n)$ so that $\rho^1(t) \ge 0$, $\int_{-\infty}^{\infty} \rho^1(t)\,dt = 1$, $\operatorname{supp}\rho^1 \subset \{t \in \mathbf{R};\ 0 \le t \le 1\}$, $\rho^n(x) \ge 0$, $\int_{\mathbf{R}^n} \rho^n(x)\,dx = 1$, $\operatorname{supp}\rho^n \subset \{x \in \mathbf{R}^n;\ |x| \le 1\}$. Here we say that $f(x) \in \mathcal{E}^{\{\kappa\}}(\mathbf{R}^n)$ if for any $T > 0$ there are $h > 0$ and $C_T > 0$ satisfying

$$|\partial_x^\alpha f(x)| \le C_T h^{|\alpha|}(|\alpha|!)^\kappa \quad \text{for } \alpha \in (\mathbf{Z}_+)^n \text{ and } x \in \mathbf{R}^n \text{ with } |x| \le T.$$

For $\varepsilon > 0$ we define

$$F_{R,\varepsilon}(t,x) = \int_{\mathbf{R}^{n+1}} \rho_\varepsilon^1(t-s)\rho_\varepsilon^n(x-y)F_R(s,y)\,dsdy,$$

for $(t,x) \in \mathbf{R}^{n+1}$, where $\rho_\varepsilon^1(t) = \varepsilon^{-1}\rho^1(t/\varepsilon)$ and $\rho_\varepsilon^n(x) = \varepsilon^{-n}\rho^n(x/\varepsilon)$. Then we have $F_{R,\varepsilon}(t,x) \in \mathcal{E}^{\{\kappa_0\}}(\mathbf{R}^{n+1})$ and

$$\operatorname{supp} F_{R,\varepsilon}(t,x) \subset \{(t,x) \in \mathbf{R}^{n+1};\ t \ge 0 \text{ and } |x| \le R + 2 + \varepsilon\}.$$

Moreover, we have

$$F_{R,\varepsilon}(t,x) \to F_R(t,x) \quad \text{in } C^\infty(\mathbf{R}; C_0^\infty(\mathbf{R}^n)) \text{ as } \varepsilon \downarrow 0$$

It follows from [8] that the Cauchy problem

$$(\mathrm{CP})_{R,\varepsilon} \qquad \begin{cases} P(t,D_t,D_x;\varepsilon)v_{R,\varepsilon}(t,x) = F_{R,\varepsilon}(t,x) & \text{in } \mathbf{R}^{n+1}, \\ v_{R,\varepsilon}(t,x)|_{t<0} = 0 \end{cases}$$

has a unique solution $v_{R,\varepsilon}(t,x)$ in $\mathcal{E}^{\{\kappa_0\}}(\mathbf{R}^{n+1})$ and that $(t_0,x^0) \notin \operatorname{supp} v_{R,\varepsilon}$ if $\operatorname{supp} F_{R,\varepsilon} \cap K^-_{(t_0,x^0)} = \emptyset$. More precisely, we have

$$\operatorname{supp} v_{R,\varepsilon} \subset \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\ (t,x) \in K^+_{(s,y)} \text{ for some } (s,y) \in \operatorname{supp} F_{R,\varepsilon}\}.$$

For $\varepsilon, \varepsilon' \in (0,1]$ with $\varepsilon' \le \varepsilon$ we put $w_{R,\varepsilon,\varepsilon'}(t,x) = v_{R,\varepsilon}(t,x) - v_{R,\varepsilon'}(t,x)$. Then we have

$$P(t,D_t,D_x;\varepsilon)w_{R,\varepsilon,\varepsilon'}(t,x) = F_{R,\varepsilon}(t,x) - F_{R,\varepsilon'}(t,x)$$

$$+\sum_{j=3}^{m}\sum_{|\alpha|\le j-3}(a_{j,\alpha}(t;\varepsilon')-a_{j,\alpha}(t;\varepsilon))D_t^{m-j}D_x^{\alpha}v_{R,\varepsilon'}(t,x).$$

Applying Lemma 2.12 we can see that there are $C_\mu>0$ ($\mu\ge m$) satisfying

$$\begin{aligned}(2.53)\quad &\sum_{l=0}^{\mu}\|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-l}D_t^l w_{R,\varepsilon,\varepsilon'}(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}\\ &\le C_\mu\Big\{\sum_{l=0}^{\mu-m}\|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-m-l+\nu_0}D_t^l(F_{R,\varepsilon}(t,x)-F_{R,\varepsilon'}(t,x))\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}\\ &\quad+\sup_{\substack{t\in[0,\delta_1],\,3\le j\le m\\ |\beta|\le j-3,\,h\le\mu-m}}|D_t^h(a_{j,\beta}(t;\varepsilon')-a_{j,\beta}(t;\varepsilon))|^2\\ &\quad\times\sum_{l=0}^{\mu-m}\|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-m-l-3+2\nu_0}D_t^lF_{R,\varepsilon'}(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}\Big\}\end{aligned}$$

for $\mu\in\mathbf{N}$ with $\mu\ge m$ and $\kappa\in\mathbf{R}$. Indeed, we also apllied Lemma 2.12 to $v_{R,\varepsilon'}(t,x)$ in order to obtain (2.53). (2.53) yields

$$\begin{aligned}&v_{R,\varepsilon}(t,x)\to v_R(t,x)\quad\text{in } C^\infty([0,\delta_1];H^\infty(\mathbf{R}^n))\text{ as }\varepsilon\downarrow 0,\\ &\operatorname{supp}v_R\cap(-\infty,\delta_1]\\ &\subset\{(t,x)\in[0,\delta_1]\times\mathbf{R}^n;\ (t,x)\in K^+_{(s,y)}\text{ for some }(s,y)\in\operatorname{supp}F_R\}.\end{aligned}$$

Since

$$\operatorname{supp}[P,\Theta(|x|-R)]u(t,x)\subset\{(t,x)\in[0,\infty)\times\mathbf{R}^n;\ R+\frac{3}{2}\le|x|\le R+2\},$$

we have

$$K^-_{(t_0,x^0)}\cap\operatorname{supp}F_R=\emptyset,$$

which proves $(t_0,x^0)\notin\operatorname{supp}v_R$ and the lemma. □

For $f(t,x)\in C^\infty(\mathbf{R}^{n+1})$ with $f(t,x)|_{t<0}=0$ we consider

$$(\mathrm{CP})_0'\qquad\begin{cases}P(t,D_t,D_x)u(t,x)=f(t,x)\quad\text{in }(-\infty,\delta_1]\times\mathbf{R}^n,\\ u(t,x)|_{t<0}=0.\end{cases}$$

Put $f_R(t,x)=\Theta(|x|-R)f(t,x)$ for $R>0$, and let $u_R(t,x)$ be a solution to $(\mathrm{CP})_0'$ with $f(t,x)$ replaced by $f_R(t,x)$. Then we have

$$P(t,D_t,D_x)(u_{R'}(t,x)-u_R(t,x))=(\Theta(|x|-R')-\Theta(|x|-R))f(t,x),$$

where $R' \geq R > 0$. Define

$$M_{\delta_1} = \sup_{\substack{t\in[0,\delta_1],\,1\leq j\leq m\\ \xi\in S^{n-1}}} |\lambda_j(t,\xi)|,$$
$$K_{\delta_1} = \{(t,x) \in \mathbf{R}^{n+1};\ t \geq |x|/M_{\delta_1}\}.$$

It is easy to see that

$$K^+_{(t_0,x^0)} \cap [0,\delta_1]\times \mathbf{R}^n \subset \{(t_0,x^0)\} + K_{\delta_1}.$$

Lemma 2.13 implies that

$$u_{R+\delta_1 M_{\delta_1}}(t,x) = u_{R'+\delta_1 M_{\delta_1}}(t,x) \quad \text{if } t \leq \delta_1 \text{ and } |x| \leq R \leq R'.$$

Therefore, we can define $u(t,x)$ by

$$u(t,x) = u_{R+\delta_1 M_{\delta_1}}(t,x) \quad \text{for } t \leq \delta_1 \text{ and } |x| \leq R,$$

and $u(t,x)\,(\in C^\infty((-\infty,\delta_1]\times\mathbf{R}^n))$ satisfies $(\mathrm{CP})_0'$. Repeating the same argument as at the end of §2.3 of [12], we can construct solutions to the Cauchy problem (CP) with $[0,\infty)\times\mathbf{R}^n$ replaced by $[0,\delta_1]\times\mathbf{R}^n$ when $f(t,x) \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j(x) \in C^\infty(\mathbf{R}^n)$ ($0 \leq j \leq m-1$), and finally we can complete the proof of Theorem 1.2.

3. Proof of Theorems 1.3 and 1.4

In this section we assume that the conditions (A), (H)′ and (T) are satisfied, and we shall give the proofs of Theorems 1.3 and 1.4, applying the arguments as in [4] (see, also, [13]).

3.1. Preliminaries

Fix j so that $1 \leq j \leq N_0$. For $(t,\xi) \in [0,\delta_1]\times(\overline{\Gamma}_j\setminus\{0\})$ we write

$$\hat{p}^{j,k}(t,\tau,\xi) = p^{j,k}(t,\tau - a_1^{j,k}(t,\xi)/3,\xi) = \tau^3 - \hat{a}_2^{j,k}(t,\xi)\tau + \hat{a}_3^{j,k}(t,\xi)$$
$$\text{if } 1 \leq k \leq r(j) \text{ and } m(j,k) = 3,$$
$$\hat{p}^{j,k}(t,\tau,\xi) = p^{j,k}(t,\tau - a_1^{j,k}(t,\xi)/2,\xi) = \tau^2 - \hat{a}_2^{j,k}(t,\xi)$$
$$\text{if } 1 \leq k \leq r(j) \text{ and } m(j,k) = 2.$$

Then we have

$$\hat{a}_2^{j,k}(t,\xi) = a_1^{j,k}(t,\xi)^2/3 - a_2^{j,k}(t,\xi)\ (\geq 0),$$

$$
\begin{aligned}
\hat{a}_3^{j,k}(t,\xi) &= 2a_1^{j,k}(t,\xi)^3/27 - a_1^{j,k}(t,\xi)a_2^{j,k}(t,\xi)/3 + a_3^{j,k}(t,\xi) \\
&\qquad\qquad \text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 3, \\
\hat{a}_2^{j,k}(t,\xi) &= a_1^{j,k}(t,\xi)^2/4 - a_2^{j,k}(t,\xi) \\
&\qquad\qquad \text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 2.
\end{aligned}
$$

Until the end of the proof of Lemma 3.3 we omit the subscript j and the superscript j of Γ_j, $P^{j,k}(\cdot)$, $p^{j,k}(\cdot)$, $\cdots$, and "j" of $r(j)$, $m(j,k)$, $\cdots$ and so forth, again. Namely, we write Γ_j, $P^{j,k}(\cdot)$, $p^{j,k}(\cdot)$, $r(j)$, $m(j,k)$, $\cdots$ as Γ, $P^k(\cdot)$, $p^k(\cdot)$, r, $m(k)$, $\cdots$, respectively. By (2.3) and the factorization theorem we have

$$
P(t,\tau,\xi) = P^1(t,\tau,\xi) \circ P^2(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi) + R(t,\tau,\xi)
$$

for $(t,\xi) \in [0,\delta_1] \times \overline{\Gamma}$ with $|\xi| \ge 1$, where

$$
P^k(t,\tau,\xi) = p^k(t,\tau,\xi) + q_0^k(t,\tau,\xi) + q_1^k(t,\tau,\xi) + r^k(t,\tau,\xi),
$$

$q_l^k(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m(k)-1,-l}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))$ ($l = 0,1$) are positively homogeneous of degree $(m(k)-1-l)$ in (τ,ξ) for $|\xi| \ge 1$ and $r^k(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m(k)-1,-2}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))$ ($1 \le k \le r$) and $R(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m-1,-\infty}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))$. Moreover, the $r^k(t,\tau,\xi)$ are classical symbols, *i.e.*, there are symbols $r_l^k(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m(k)-1,-2-l}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))$ ($l \in \mathbf{Z}_+$) such that the $r_l^k(t,\tau,\xi)$ are positively homogeneous of degree $(m(k)-3-l)$ in (τ,ξ) for $|\xi| \ge 1$ and

$$
r^k(t,\tau,\xi) - \sum_{l=0}^{N-1} r_l^k(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m(k)-1,-2-N}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\})) \qquad (N = 1,2,\cdots). \tag{3.1}
$$

We write

$$
r^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} r_l^k(t,\tau,\xi) \quad \text{in } \mathcal{S}_{1,0}^{m(k)-1,-2}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\}))
$$

if (3.1) is valid. We also write

$$
\begin{aligned}
\mathcal{S}_{cl}^{m,\nu}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\})) =& \{a(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m,\nu}([0,\delta_1]\times(\overline{\Gamma}\setminus\{0\})); \\
&\qquad a(t,\tau,\xi) \text{ is a classical symbol}\}.
\end{aligned}
$$

Define

$$
p^k(t,\tau,\xi) = (-1)^{m(k)} p^k(t,-\tau,-\xi),
$$

$$q_l^k(t,\tau,\xi) = (-1)^{m(k)-1-l} q_l^k(t,-\tau,-\xi) \quad (\ l=0,1)$$
$$\text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (-\overline{\Gamma} \setminus \{0\}).$$

Moreover, we define $r^k(t,\tau,\xi) \in \mathcal{S}_{cl}^{m(k)-1,-2}([0,\delta_1] \times (-\overline{\Gamma} \setminus \{0\}))$ so that

$$r^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} (-1)^{m(k)-3-l} r_l^k(t,-\tau,-\xi) \qquad \text{in } \mathcal{S}_{1,0}^{m(k)-1,-2}([0,\delta_1] \times (-\overline{\Gamma} \setminus \{0\})). \tag{3.2}$$

In fact, we can easily construct a symbol $r^k(t,\tau,\xi)$ for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (-\overline{\Gamma} \setminus \{0\})$ satisfying (3.2). Note that $r^k(t,\tau,\xi)$ is uniquely determined modulo $\mathcal{S}_{1,0}^{m(k)-1,-\infty}([0,\delta_1] \times (-\overline{\Gamma} \setminus \{0\}))$. Put

$$P^k(t,\tau,\xi) = p^k(t,\tau,\xi) + q_0^k(t,\tau,\xi) + q_1^k(t,\tau,\xi) + r^k(t,\tau,\xi)$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((\overline{\Gamma} \cup (-\overline{\Gamma})) \setminus \{0\})$. Then we have the following

LEMMA **3.1.** *We have*

$$P(t,\tau,\xi) \equiv P^1(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi) \quad (\text{mod } \mathcal{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times ((\overline{\Gamma} \cup (-\overline{\Gamma})) \setminus \{0\}))).$$

PROOF. Write

$$P^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} P_l^k(t,\tau,\xi) \quad \text{in } \mathcal{S}_{1,0}^{m(k)}([0,\delta_1] \times ((\overline{\Gamma} \cup (-\overline{\Gamma})) \setminus \{0\})) \quad (\ 1 \le k \le r),$$

where the $P_l^k(t,\tau,\xi)$ are positively homogeneous of degree $(m(k)-l)$ in (τ,ξ). We also write

$$P^1(t,\tau,\xi) \circ P^2(t,\tau,\xi) \circ \cdots \circ P^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} I_l^{1,2,\cdots,k}(t,\tau,\xi) \quad \text{in } \mathcal{S}_{1,0}^{m(1)+\cdots+m(k)}([0,\delta_1] \times ((\overline{\Gamma} \cup (-\overline{\Gamma})) \setminus \{0\})) \ (\ 1 \le k \le r),$$

where the $I_l^{1,2,\cdots,k}(t,\tau,\xi)$ are positively homogeneous of degree $(m(1)+\cdots+m(k)-l)$. For example, the $I_l^{1,2}(t,\tau,\xi)$ are given by

$$I_l^{1,2}(t,\tau,\xi) = \sum_{\substack{h,\mu,\nu \in \mathbf{Z}_+ \\ h+\mu+\nu=l}} \frac{1}{h!} \partial_\tau^h P_\mu^1(t,\tau,\xi) \cdot D_t^h P_\nu^2(t,\tau,\xi).$$

Then it is easy to see that

$$
\begin{aligned}
I_l^{1,2}(t,\tau,\xi) &= \sum_{\substack{h,\mu,\nu\in\mathbf{Z}_+\\ h+\mu+\nu=l}} (-1)^{m(1)+m(2)-l}\frac{1}{h!}(\partial_\tau^h P_\mu^1)(t,-\tau,-\xi)(D_t^h P_\nu^2)(t,-\tau,-\xi)\\
&= (-1)^{m(1)+m(2)-l} I_l^{1,2}(t,-\tau,-\xi)\\
&\qquad\qquad\qquad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times((-\overline{\Gamma})\setminus\{0\}).
\end{aligned}
$$

Moreover, we can prove by induction on k that

$$
\begin{aligned}
(3.3)\qquad I_l^{1,\cdots,k}(t,\tau,\xi) &= (-1)^{m(1)+\cdots m(k)-l} I_l^{1,\cdots,k}(t,-\tau,-\xi)\\
&\qquad \text{for } 2\le k\le r \text{ and } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times((\overline{\Gamma})\setminus\{0\}).
\end{aligned}
$$

Since $P(t,\tau,\xi)$ is a polynomial of (τ,ξ) and

$$
P(t,\tau,\xi) - P^1(t,\tau,\xi)\circ\cdots\circ P^r(t,\tau,\xi)\in \mathcal{S}_{1,0}^{m-1,-\infty}([0,\delta_1]\times(-\overline{\Gamma})\setminus\{0\})),
$$

(3.3) proves the lemma. □

We write

$$
\begin{aligned}
q_0^k(t,\tau,\xi) &= b_0^k(t,\xi)\tau^2 + b_1^k(t,\xi)\tau + b_2^k(t,\xi),\\
q_0^k(t,\tau-a_1^k(t,\xi)/3,\xi) &= \hat{b}_0^k(t,\xi)\tau^2 + \hat{b}_1^k(t,\xi)\tau + \hat{b}_2^k(t,\xi),
\end{aligned}
$$

if $1\le k\le r$ and $m(k)=3$. Then it is obvious that

$$
\begin{aligned}
\hat{b}_0^k(t,\xi) &= b_0^k(t,\xi)\\
\hat{b}_1^k(t,\xi) &= b_1^k(t,\xi) - \frac{2}{3}a_1^k(t,\xi)b_0^k(t,\xi),\\
\hat{b}_2^k(t,\xi) &= b_2^k(t,\xi) + \frac{1}{9}a_1^k(t,\xi)^2 b_0^k(t,\xi) - \frac{1}{3}a_1^k(t,\xi)b_1^k(t,\xi).
\end{aligned}
$$

LEMMA **3.2.** *Assume that $1\le k\le r$ and $m(k)=3$. Putting $b(t,\tau,\xi)=$ sub $\sigma(P^k)(t,\tau-a_1^k(t,\xi)/3,\xi)$ we have the following:*
(i) *There is $C_1>0$ satisfying*

$$
\begin{aligned}
(3.4)\qquad \min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|b(t,\tau,\xi)| &\le C_1 h_2(t,\tau,\xi;\hat{p}^k)^{1/2}\\
&\qquad \textit{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\overline{\Gamma}\cap S^{n-1})
\end{aligned}
$$

if and only if there is $C_2>0$ satisfying

$$
(3.5)\qquad \min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|b(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)|
$$

$$\le C_2 h_2(t,(\hat a_3^k(t,\xi)/2)^{1/3},\xi;\hat p^k)^{1/2},$$

(3.6) $$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|(\partial_\tau b)(t,(\hat a_3^k(t,\xi)/2)^{1/3},\xi)|$$

$$\le C_2 h_1(t,0,\xi;\hat p^k)^{1/2}\,(=\sqrt{2}C_2\hat a_2^k(t,\xi)^{1/2})$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}\cap S^{n-1})$.

(ii) (3.4) *is valid if and only if there is* $C_3>0$ *satisfying*

(3.7) $$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|b(t,A^k(t,\xi),\xi)|$$

$$\le C_3 h_2(t,A^k(t,\xi),\xi;\hat p^k)^{1/2},$$

(3.8) $$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|(\partial_\tau b)(t,A^k(t,\xi),\xi)|$$

$$\le C_3 h_1(t,0,\xi;\hat p^k)^{1/2}\,(=\sqrt{2}C_3\hat a_2^k(t,\xi)^{1/2})$$

for $(t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}\cap S^{n-1})$,

where

$$\nu^k(t,\xi)=\begin{cases}1 & \textit{if } \hat a_3^k(t,\xi)\ge 0,\\ -1 & \textit{if } \hat a_3^k(t,\xi)<0,\end{cases}$$

(3.9) $$A^k(t,\xi)=\nu^k(t,\xi)(\hat a_2^k(t,\xi)/3)^{1/2}.$$

Remark. Assume that $m(k)=3$. Then we have

(3.10) $$h_2(t,\tau,\xi;\hat p^k)=h_2(t,\tau-a_1^k(t,\xi)/3,\xi;p^k)$$
$$=3\tau^4+\hat a_2^k(t,\xi)^2-6\tau\hat a_3(t,\xi),$$
$$h_1(t,\tau,\xi;\hat p^k)=h_1(t,\tau-a_1^k(t,\xi)/3,\xi;p^k)=3\tau^2+2\hat a_2^k(t,\xi).$$

Hyperbolicity implies that

(3.11) $$(\hat a_3^k(t,\xi)/2)^2\le(\hat a_2^k(t,\xi)/3)^3,$$

and the discriminant $\widehat{D}^k(t,\xi)$ of $\hat p^k(t,\tau,\xi)=0$ in τ is given by

(3.12) $$\widehat{D}^k(t,\xi)\,(=D^k(t,\xi))=4\hat a_2^k(t,\xi)^3-27\hat a_3^k(t,\xi)^2,$$

where $D^k(t,\xi)$ denotes the discriminant of $p^k(t,\tau,\xi)=0$ in τ.

Proof. Write

$$\hat p^k(t,\tau,\xi)=\prod_{l=1}^{3}(\tau-\hat\lambda_l^k(t,\xi)),\quad \textit{i.e.},$$

$$\hat\lambda_l^k(t,\xi) = \lambda_l^k(t,\xi) + a_1^k(t,\xi)/3 \quad (\ 1 \le l \le 3)$$

Assume that (3.4) is valid. Then (3.5) is valid with $C_2 \ge C_1$. Fix $(t,\xi) \in [0,\delta_1] \times (\overline{\Gamma} \cap S^{n-1})$. We first consider the case where $\hat\lambda_l^k(t,\xi) \ne \hat\lambda_\mu^k(t,\xi)$ for $1 \le l < \mu \le 3$. Then we can write

$$b(t,\tau,\xi) = \sum_{l=1}^{3} b_l(t,\xi)\hat p_l^k(t,\tau,\xi), \tag{3.13}$$

where

$$b_l(t,\xi) = b(t,\hat\lambda_l^k(t,\xi),\xi)/\hat p_l^k(t,\hat\lambda_l^k(t,\xi),\xi) \quad (\ 1 \le l \le 3).$$

(3.4) gives

$$\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|b_l(t,\xi)| \le C_1.$$

By (3.13) we have

$$\partial_\tau b(t,\tau,\xi) = \sum_{l=1}^{3} b_l(t,\xi)(2\tau + \hat\lambda_l^k(t,\xi)),$$

since $\sum_{\mu=1}^3 \hat\lambda_\mu^k(t,\xi) = 0$. Therefore, we have

$$\begin{aligned} \min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|\partial_\tau b(t,\tau,\xi)| &\le C_1\Big\{6|\tau| + \sqrt{3}\Big(\sum_{l=1}^{3} \hat\lambda_l^k(t,\xi)^2\Big)^{1/2}\Big\} \\ &= C_1\{6|\tau| + \sqrt{6}\hat a_2^k(t,\xi)^{1/2}\}, \end{aligned} \tag{3.14}$$

since

$$\sum_{l=1}^{3} \hat\lambda_l^k(t,\xi)^2 = 2\hat a_2^k(t,\xi). \tag{3.15}$$

So (3.11) and (3.14) yield (3.6) with $C_2 \ge (2\sqrt{3}+\sqrt{6})C_1$. Next consider the case where $\hat\lambda_1^k(t,\xi) \ne \hat\lambda_2^k(t,\xi) = \hat\lambda_3^k(t,\xi)$, for instance. Then we have $h_2(t,\hat\lambda_2^k(t,\xi),\xi;\hat p^k) = 0$ and, therefore, we can write

$$b(t,\tau,\xi) = (\tau - \hat\lambda_2^k(t,\xi))\hat b(t,\tau,\xi), \tag{3.16}$$

where $\hat b(t,\tau,\xi)$ is a linear expression of τ. (3.4) yields

$$\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|\hat b(t,\tau,\xi)| \le C_1\{\sqrt{2}|\tau - \hat\lambda_1^k(t,\xi)| + |\tau - \hat\lambda_2^k(t,\xi)|\}. \tag{3.17}$$

So we have

$$\begin{aligned}&\hat{b}(t,\tau,\xi)=\hat{b}_1(t,\xi)(\tau-\hat{\lambda}_1^k(t,\xi))+\hat{b}_2(t,\xi)(\tau-\hat{\lambda}_2^k(t,\xi)),\\ &\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|\hat{b}_l(t,\xi)|\le\sqrt{2}C_1\quad(\ l=1,2),\end{aligned}\tag{3.18}$$

where

$$\hat{b}_l(t,\xi)=(-1)^l\hat{b}(t,\hat{\lambda}_{3-l}^k(t,\xi),\xi)/(\hat{\lambda}_1^k(t,\xi)-\hat{\lambda}_2^k(t,\xi))\quad(\ l=1,2).$$

Since

$$\partial_\tau b(t,\tau,\xi)=\hat{b}_1(t,\xi)(\tau-\hat{\lambda}_1^k(t,\xi))+(\hat{b}_1(t,\xi)+2\hat{b}_2(t,\xi))(\tau-\hat{\lambda}_2^k(t,\xi)),\tag{3.19}$$

(3.11) and (3.15) – (3.19) give

$$\begin{aligned}&\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|(\partial_\tau b)(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)|\\ &\le 4\sqrt{2}C_1\{(\hat{a}_2^k(t,\xi)/3)^{1/2}+2\hat{a}_2^k(t,\xi)^{1/2}\}\le 12\sqrt{2}C_1\hat{a}_2^k(t,\xi)^{1/2},\end{aligned}$$

which proves that (3.6) is valid. Finally consider the case where $\hat{\lambda}_1^k(t,\xi)=\hat{\lambda}_2^k(t,\xi)=\hat{\lambda}_3^k(t,\xi)\,(=0)$. Then we have $\hat{a}_2^k(t,\xi)=\hat{a}_3^k(t,\xi)=0$,

$$h_2(t,\tau,\xi;\hat{p}^k)=3\tau^4\quad\text{and}\quad h_1(t,\tau,\xi;\hat{p}^k)=3\tau^2.$$

Therefore, we can write

$$b(t,\tau,\xi)=\tau^2\hat{b}(t,\xi),\tag{3.20}$$

where

$$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|\hat{b}(t,\xi)|\le\sqrt{3}C_1.\tag{3.21}$$

This yields

$$\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|(\partial_\tau b)(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)|=0\le h_1(t,0,\xi;\hat{p}^k)^{1/2}\,(=0).$$

Next we assume that (3.5) and (3.6) are valid. Write

$$\begin{aligned}b(t,\tau,\xi)&=b(t,(\hat{a}_3^k/2)^{1/3},\xi)+(\partial_\tau b)(t,(\hat{a}_3^k/2)^{1/3},\xi)(\tau-(\hat{a}_3^k/2)^{1/3})\\ &\quad+\frac{1}{2}(\partial_\tau^2 b)(t,0,\xi)(\tau-(\hat{a}_3^k/2)^{1/3})^2,\\ h_2(t,\tau,\xi;\hat{p}^k)&=9((\hat{a}_2^k/3)^2-(\hat{a}_3^k/2)^{4/3})+6(\hat{a}_3^k/2)^{2/3}(\tau-(\hat{a}_3^k/2)^{1/3})^2\\ &\quad+3(\tau^2-(\hat{a}_3^k/2)^{2/3})^2,\end{aligned}\tag{3.22}$$

where $\hat{a}_l^k = \hat{a}_l^k(t,\xi)$ ($l = 2,3$). Since

$$h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k) = 9((\hat{a}_2/3)^2 - (\hat{a}_3/2)^{4/3}), \tag{3.23}$$

we have

$$h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k) \le h_2(t, \tau, \xi; \hat{p}^k). \tag{3.24}$$

Moreover, we have

$$\begin{aligned}
(\tau - (\hat{a}_3^k/2)^{1/3})^2 &\le |\tau^2 - (\hat{a}_3^k/2)^{2/3}| + 2|(\hat{a}_3^k/2)^{1/3}(\tau - (\hat{a}_3^k/2)^{1/3})| \\
&\le h_2(t,\tau,\xi;\hat{p}^k)^{1/2}/\sqrt{3} + 2h_2(t,\tau,\xi;\hat{p}^k)^{1/2}/\sqrt{6} \le 2h_2(t,\tau,\xi;\hat{p}^k)^{1/2},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\{(\hat{a}_2^k)^{1/2}(\tau - (\hat{a}_3^k/2)^{1/3})\}^4 &= h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k)(\tau - (\hat{a}_3^k/2)^{1/3})^4 \\
&\quad + \{3(\hat{a}_3^k/2)^{2/3}(\tau - (\hat{a}_3^k/2)^{1/3})^2\}^2 \\
&\le 5h_2(t,\tau,\xi;\hat{p}^k)^2.
\end{aligned} \tag{3.26}$$

We may assume that $|(\partial_\tau^2 b)(t,0,\xi)| \le C_2$. Therefore, (3.4) is valid with $C_1 \ge 6C_2$, which proves the assertion (i). (3.22) gives

$$\begin{aligned}
&h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k) = 9\{(\hat{a}_2^k/3)^2 - (\hat{a}_3^k/2)^{4/3}\} \\
&= 9\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\}\{(\hat{a}_2^k/3)^{1/2} + (|\hat{a}_3^k|/2)^{1/3}\} \\
&\quad \times \{(\hat{a}_2^k/3) + (|\hat{a}_3^k|/2)^{2/3}\}.
\end{aligned}$$

So we have

$$9(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \le h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k). \tag{3.27}$$

On the other hand, we have

$$\begin{aligned}
&h_2(t, A^k(t,\xi), \xi; \hat{p}^k) = 12(\hat{a}_2^k/3)^2 - 12(\hat{a}_2^k/3)^{1/2}(|\hat{a}_3^k|/2) \\
&= 12(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \\
&\quad \times \{(\hat{a}_2^k/3) + (\hat{a}_2^k/3)^{1/2}(|\hat{a}_3^k|/2)^{1/3} + (|\hat{a}_3^k|/2)^{2/3}\}.
\end{aligned}$$

By (3.11) we have

$$\begin{aligned}
&12(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \\
&\le h_2(t, A^k(t,\xi), \xi; \hat{p}^k) \le 36(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\},
\end{aligned} \tag{3.28}$$

which, with (3.24) and (3.27), yields

$$h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k) \le h_2(t, A^k(t,\xi), \xi; \hat{p}^k) \le 4h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k). \tag{3.29}$$

Now we can prove the assertion (ii). Note that

$$\partial_t \partial_\tau^2 p^k(t,\tau,\xi) = 2\partial_t a_1^k(t,\xi),$$
$$\partial_\tau b(t,\tau,\xi) = 2b_0^k(t,\xi)\tau + \hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi).$$

So we have

$$\begin{aligned}(3.30)\qquad &|(\partial_\tau b)(t, A^k(t,\xi),\xi) - (\partial_\tau b)(t, (\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| \\ &= 2|b_0(t,\xi)|\{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\} \\ &\le 2|b_0(t,\xi)|(\hat{a}_2^k(t,\xi)/3)^{1/2}\end{aligned}$$

since $|A^k(t,\xi) - (\hat{a}_3^k(t,\xi)/2)^{1/3}| = (\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}$. This implies that (3.6) is valid if and only if (3.8) is valid. We have also

$$\begin{aligned}(3.31)\qquad &|b(t, A^k(t,\xi),\xi) - b(t, (\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| \\ &\le \{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\}|(\partial_\tau b)(t, (\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| \\ &\quad + \{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\}^2 |(\partial_\tau^2 b)(t,0,\xi)|/2.\end{aligned}$$

It follows from (3.23) and (3.24) that

$$\begin{aligned}(3.32)\qquad &3(\hat{a}_2^k(t,\xi)/3)^{1/2}\{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\} \\ &\le h_2(t, (\hat{a}_3^k(t,\xi)/2)^{1/3}, \xi; \hat{p}^k)^{1/2} \le h_2(t, A^k(t,\xi),\xi;\hat{p}^k)^{1/2},\end{aligned}$$

since $(\alpha-\beta)^{1/2} \ge \alpha^{1/2} - \beta^{1/2}$ if $\alpha \ge \beta \ge 0$. This, together with (3.6) and (3.31), proves that (3.5) and (3.6) hold if and only if (3.7) and (3.8) hold. □

LEMMA **3.3.** *Assume that* $1 \le k \le r$, *and* $m(k) = 2$. *Then there is* $C_1 > 0$ *satisfying*

$$(3.33)\qquad \min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|sub\ \sigma(P^k)(t,\tau,\xi)| \le C_1 h_1(t,\tau,\xi;p^k)^{1/2}$$
$$\textit{for } (t,\tau,\xi) \in [0,\delta_1]\times \mathbf{R} \times (\overline{\Gamma}\cap S^{n-1})$$

if and only if there is $C_2 > 0$ *satisfying*

$$\begin{aligned}(3.34)\qquad &\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|sub\ \sigma(P^k)(t, -a_1^k(t,\xi)/2,\xi)| \\ &\le C_2 h_1(t, -a_1^k(t,\xi)/2,\xi;p^k)^{1/2} \quad \textit{for } (t,\xi) \in [0,\delta_1]\times(\overline{\Gamma}\cap S^{n-1}).\end{aligned}$$

REMARK. Assume that $m(k) = 2$. Then we have

$$(3.35)\qquad h_1(t,\tau,\xi;p^k) = 2(\tau + a_1^k(t,\xi)/2)^2 + 2\hat{a}_2^k(t,\xi).$$

PROOF. We have

$$
\begin{aligned}
&sub\ \sigma(P^k)(t,\tau,\xi) = sub\ \sigma(P^k)(t,-a_1^k(t,\xi)/2,\xi)\\
&\quad + (\partial_\tau sub\ \sigma(P^k))(t,0,\xi)(\tau + a_1^k(t,\xi)/2).
\end{aligned}
$$

Therefore, this, together with (3.35), proves the lemma. □

Define

$$
\beta^k(t,\xi) = sub\ \sigma(P^k)(t, A^k(t,\xi) - a_1^k(t,\xi)/3,\xi) \tag{3.36}
$$

for $1 \le k \le r$ with $m(k) = 3$. We note that

$$
\begin{aligned}
&\partial_t a_2^k(t,\xi) = 2a_1^k(t,\xi)\partial_t a_1^k(t,\xi)/3 - \partial_t \hat{a}_2^k(t,\xi),\\
&sub\ \sigma(P^k)(t,\tau - a_1^k(t,\xi)/3,\xi) \qquad (3.37)\\
&= q_0^k(t,\tau - a_1^k(t,\xi)/3,\xi) + i\partial_t a_1^k(t,\xi)\cdot\tau - \frac{i}{2}\partial_t \hat{a}_2^k(t,\xi)
\end{aligned}
$$

if $1 \le k \le r$ and $m(k) = 3$.

LEMMA **3.4.** *Let* $k \in \mathbf{N}$ *satisfy* $1 \le k \le r$ *and* $m(k) = 3$. (i) *Assume that* $\hat{a}_2^k(t,\xi) \not\equiv 0$ *in* (t,ξ). *Then there is* $C_1 > 0$ *satisfying*

$$
\begin{aligned}
&\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|sub\ \sigma(P^k)(t,\tau - a_1^k(t,\xi)/3,\xi)| \qquad (3.38)\\
&\le C_1 h_2(t,\tau,\xi;\hat{p}^k)^{1/2} \quad \text{for } (t,\tau,\xi) \in [0,\delta_1]\times\mathbf{R}\times(\overline{\Gamma}\cap S^{n-1})
\end{aligned}
$$

if and only if there is $C_2 > 0$ *satisfying*

$$
\begin{aligned}
&\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|\beta^k(t,\xi)|\hat{a}_2^k(t,\xi) \qquad (3.39)\\
&\le C_2(\widehat{D}^k(t,\xi)\hat{a}_2^k(t,\xi))^{1/2},\\
&\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\}|\hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi)| \qquad (3.40)\\
&\le C_2\hat{a}_2^k(t,\xi)^{1/2} \quad \text{for } (t,\xi) \in [0,\delta_1]\times(\overline{\Gamma}\cap S^{n-1}).
\end{aligned}
$$

(ii) *Assume that* $\hat{a}_2^k(t,\xi) \equiv 0$. *Then* (3.38) *is valid if and only if*

$$
\hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi) = \hat{b}_2^k(t,\xi) = 0 \quad \text{for } (t,\xi) \in [0,\delta_1]\times(\overline{\Gamma}\cap S^{n-1}).
$$

PROOF. Assume that $\hat{a}_2(t,\xi) \not\equiv 0$ in (t,ξ). By virtue of Lemma 3.2 it is enough to prove that the conditions (3.7) and (3.8) are equivalent to the conditions (3.39) and (3.40). Here we may modify the constants appropriately. By (3.37) we see that (3.8) is valid if and only if (3.40) is valid. (3.11) and (3.12) yield

$$
108(\hat{a}_2^k/3)^2\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\} \le \widehat{D}^k(t,\xi) \tag{3.41}
$$

$$
\begin{aligned}
&= 108\{(\hat{a}_2^k/3)^3 - (\hat{a}_3^k/2)^2\} = 108\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\} \\
&\quad \times \{(\hat{a}_2^k/3)^2 + (\hat{a}_2^k/3)(|\hat{a}_3^k|/2)^{2/3} + (|\hat{a}_3^k|/2)^{4/3}\} \\
&\leq 324(\hat{a}_2^k/3)^2\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\},
\end{aligned}
$$

where $\hat{a}_l^k = \hat{a}_l^k(t,\xi)$ ($l = 2, 3$). This, together with (3.28), yields

$$
\begin{aligned}
3(\hat{a}_2^k/3)h_2(t, A^k(t,\xi), \xi; \hat{p}^k) &\leq \widehat{D}^k(t,\xi) \\
&\leq 54(\hat{a}_2^k/3)h_2(t, A^k(t,\xi), \xi; \hat{p}^k),
\end{aligned}
\tag{3.42}
$$

since

$$
\begin{aligned}
&(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \leq (\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3} \\
&\leq 2(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\}.
\end{aligned}
$$

Therefore, if (3.7) is valid, then (3.39) is valid with $C_2 = \sqrt{3}C_3$. Applying the Weierstrass preparation theorem to $\hat{a}_2^k(t,\xi)$, we can prove that (3.7) is valid with $C_3 = 6C_2$ if (3.39) is valid, which proves the assertion (i). Next assume that $\hat{a}_2^k(t,\xi) \equiv 0$ in (t,ξ). Then, by (3.10) and (3.11) we have

$$
\hat{a}_3(t,\xi) \equiv \widehat{D}^k(t,\xi) \equiv 0 \quad \text{and} \quad h_2(t,\tau,\xi;\hat{p}^k) \equiv 3\tau^4,
$$

which proves the assertion (ii). □

For $(t_0, x^0) \in (0, \delta_1] \times \mathbf{R}^n$ and $\varepsilon > 0$ we put

$$
\Omega_\varepsilon(t_0, x^0) = \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\ t_0 - t > \varepsilon|x - x^0|^2\}.
$$

Lemma **3.5.** *Assume that the Cauchy problem* (CP) *is C^∞ well-posed and has finite propagation property. Then there is $\varepsilon_0 > 0$ such that for $(t_0, x^0) \in [0,\infty) \times \mathbf{R}^n$ and $p \in \mathbf{Z}_+$ there are $C > 0$ and $q \in \mathbf{Z}_+$ satisfying*

$$
|u|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} \leq C|Pu|_{q,\Omega_{\varepsilon_0}(t_0,x^0)}
$$

for any $u \in C^\infty(\mathbf{R}^{n+1})$ with $u(t,x)|_{t<0} = 0$. Here $|f|_{p,K}$ is defined by

$$
|f|_{p,K} = \sup_{(t,x)\in K,\, j+|\alpha|\leq p} |D_t^j D_x^\alpha f(t,x)|.
$$

Proof. We can choose $\varepsilon_0 > 0$ so that

$$
(\{(t_1, x^1)\} - \Gamma_0) \cap \{t \geq 0\} \subset \Omega_{\varepsilon_0}(t_0, x^0)
$$

if $(t_0, x^0) \in (0, \delta_1] \times \mathbf{R}^n$, $(t_1, x^1) \in \Omega_\varepsilon(t_0, x^0)$ and $t_1 \geq 0$. Here Γ_0 is a proper convex closed cone in $\mathbf{R}^{n+1}$ such that $\Gamma_0 \subset \{t > 0\} \cup \{0\}$ and Γ_0 satisfies the following:

$$
u(t,x) = 0 \quad \text{in } \Gamma_0(t_0,x^0)\ (\equiv \{(t_0,x^0)\} - \Gamma_0)
$$

$$\text{if } (t_0, x^0) \in [0, \delta_1] \times \mathbf{R}^n,\ u \in C^\infty(\mathbf{R}^{n+1}),$$
$$\operatorname{supp} u \subset \{t \ge 0\} \text{ and } P(t, D_t, D_x)u(t,x) = 0 \text{ in } \Gamma_0(t_0, x^0).$$

Define

$$X = \{f \in C^\infty(\mathbf{R}^{n+1});\ \operatorname{supp} f \subset \{t \ge 0\}\}.$$

X is a closed subspace of the Fréchet space $C^\infty(\mathbf{R}^{n+1})$. The operator $X \ni f(t,x) \mapsto u(t,x) \in X$ is a closed operator, where $u(t,x)$ is a unique solution in X satisfying $Pu(t,x) = f(t,x)$. So Banach's closed graph theorem proves that for any compact subset K of $[0,\infty) \times \mathbf{R}^n$ and $p \in \mathbf{Z}_+$, there are a compact subset K' of $[0,\infty) \times \mathbf{R}^n$, $C_{p,K} > 0$ and $q \in \mathbf{Z}_+$ satisfying

$$|u|_{p,K} \le C_{p,K} |Pu|_{q,K'} \quad \text{for } u \in X. \tag{3.43}$$

It follows from [2] that for any $u \in X$ and $(t_0, x^0) \in [0, \delta_1] \times \mathbf{R}^n$ there are $f \in X$ and $C' > 0$ such that $f = Pu$ in $\Omega_{\varepsilon_0}(t_0.x^0)$ and

$$|f|_{q,\mathbf{R}^{n+1}} \le C' |Pu|_{q,\Omega_{\varepsilon_0}(t_0,x^0)} \tag{3.44}$$

(see, also, [6]). By the assumptions there is $v \in X$ satisfying $Pv = f$. Then finite propagation property implies that $v(t,x) = u(t,x)$ in $\Omega_{\varepsilon_0}(t_0, x^0)$. (3.43) with $K = \overline{\Omega_{\varepsilon_0}(t_0, x^0)} \cap \{t \ge 0\}$ and (3.44) yield

$$\begin{aligned} |u|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} &= |v|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} \le C_{p,K} |f|_{q,K'} \le C_{p,K} |f|_{q,\mathbf{R}^{n+1}} \\ &\le C' C_{p,K} |Pu|_{p,\Omega_{\varepsilon_0}(t_0,x^0)}, \end{aligned}$$

which proves the lemma. □

3.2. The triple characteristic factors

We factorized $p(t,\tau,\xi)$ as (2.3):

$$p(t,\tau,\xi) = \prod_{k=1}^{r(j)} p^{j,k}(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1}),$$

where $1 \le j \le N_0$. In this subsection we omit the subscript j and the superscript j, and "j" of $r(j)$ and $m(j,k)$, in the same manner as in §3.1. Fix $k_0 \in \mathbf{N}$ so that $1 \le k_0 \le r$ and $m(k_0) = 3$. We also define $D_l^{k_0}(t,\xi)$ ($0 \le l \le 3$) by

$$\tau^3 + \sum_{l=1}^{3} D_l^{k_0}(t,\xi)\tau^{3-l} = \prod_{1 \le k < l \le 3} (\tau + \mu_{k,l}^{k_0}(t,\xi)),$$

$$D_0^{k_0}(t,\xi) \equiv 1,$$

where $\mu_{k,l}^{k_0}(t,\xi) = (\lambda_k^{k_0}(t,\xi) - \lambda_l^{k_0}(t,\xi))^2$. Then we have

$$\begin{aligned} D_3^{k_0}(t,\xi) &= \widehat{D}^{k_0}(t,\xi) = 4\hat{a}_2^{k_0}(t,\xi)^3 - 27\hat{a}_3^{k_0}(t,\xi)^2, \\ D_2^{k_0}(t,\xi) &= 9\hat{a}_2^{k_0}(t,\xi)^2, \\ D_1^{k_0}(t,\xi) &= 6\hat{a}_2^{k_0}(t,\xi). \end{aligned}$$

By the factorization theorem we can write

$$\begin{aligned} P(t,\tau,\xi) = &P^1(t,\tau,\xi) \circ \cdots \circ P^{k_0-1}(t,\tau,\xi) \circ P^{k_0+1}(t,\tau,\xi) \circ \\ &\cdots \circ P^r(t,\tau,\xi) \circ P^{k_0}(t,\tau,\xi) + R(t,\tau,\xi), \end{aligned} \tag{3.45}$$

where $R(t,\tau,\xi) \in \mathcal{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times (\overline{\Gamma} \setminus \{0\}))$. We note that the $P^k(t,\tau,\xi)$ are different from the $P^{j,k}(t,\tau,\xi)$ in (2.4) with $\varepsilon = 0$ if $k_0 \neq r$, and that whether (2.18) and (2.19) are satisfied or not does not depend on the order of the product in (2.4) (see Lemma 2.5 and its remark). We may assume that $P^k(t,\tau,\xi)$ are defined for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((\overline{\Gamma} \cup (-\overline{\Gamma})) \setminus \{0\})$ as stated in §3.1. For $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((-\overline{\Gamma}) \setminus \{0\})$ we define $R(t,\tau,\xi)$ by

$$\begin{aligned} R(t,\tau,\xi) = P(t,\tau,\xi) - P^1(t,\tau,\xi) \circ \cdots \circ P^{k_0-1}(t,\tau,\xi) \\ \circ P^{k_0+1}(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi) \circ P^{k_0}(t,\tau,\xi) \end{aligned}$$

(see Lemma 3.1). Now fix k_0, and write $P^{k_0}(t,\tau,\xi)$, $p^{k_0}(t,\tau,\xi)$, $D_l^{k_0}(t,\xi)$, $\cdots$ as $P(t,\tau,\xi)$, $p(t,\tau,\xi)$, $D_l(t,\xi)$, $\cdots$, *i.e.*,

$$\begin{aligned} p(t,\tau,\xi) &= \tau^3 + a_1(t,\xi)\tau^2 + a_2(t,\xi)\tau + a_3(t,\xi), \\ \hat{p}(t,\tau,\xi) &= p(t,\tau - a_1(t,\xi)/3,\xi) = \tau^3 - \hat{a}_2(t,\xi)\tau + \hat{a}_3(t,\xi), \\ P(t,\tau,\xi) &= p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi) + r(t,\tau,\xi) \end{aligned}$$

until Lemma 3.10, where $q_l(t,\tau,\xi) \in \mathcal{S}_{1,0}^{2,-l}([0,\delta_1] \times (\overline{\Gamma} \setminus \{0\}))$ is positively homogeneous of degree $(2-l)$ in (τ,ξ) for $|\xi| \geq 1$ ($l = 0,1$) and $r(t,\tau,\xi) \in \mathcal{S}_{1,0}^{2,-2}([0,\delta_1] \times (\overline{\Gamma} \setminus \{0\}))$. Let $t_0 \in [0,\delta_1/2]$, $\xi^0 \in \Gamma \cap S^{n-1}$ and $\theta_0 > 0$, and let $T(\theta), \Xi_l(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ ($1 \leq l \leq n$) be real-valued functions satisfying the following:

(i) $0 < t_0 + T(\theta) \leq \delta_1$ for $\theta \in (0,\theta_0]$.

(ii) $T(0) = 0$ and $\Xi(0) = \xi^0$, where $\Xi(\theta) = (\Xi_1(\theta), \cdots, \Xi_n(\theta))$.

(iii) $\Xi(\theta) \in S^{n-1}$ for $\theta \in [0,\theta_0]$ and the $\Xi_l(\theta)$ are real analytic in $[0,\theta_0]$.

(iv) $T(\theta)$ can be expanded into a convergent Puiseux series of $\theta \in [0, \theta_0]$.

We say that $T(\theta)$ and $\Xi(\theta)$ satisfy the condition (T, Ξ) if the above conditions (i) – (iv) are satisfied.

(I) The case where $D_3(t, \Xi(\theta)) \not\equiv 0$ in (t, θ).

Applying the Weierstrass preparation theorem, we can write

$$\begin{aligned} D_3(t_0+t, \Xi(\theta))\hat{a}_2(t_0+t, \Xi(\theta)) &= \sum_{l=l_0}^{\infty} d_l(t)\theta^l \\ &= \theta^{l_0} d(t,\theta) \prod_{i=1}^{n_0} (t - t_i(\theta)), \quad d(t,\theta) \neq 0 \end{aligned}$$

for $(t, \theta) \in [-\delta_0, \delta_0] \times [0, \theta_0]$, where $0 < \delta_0 \le \delta_1 - t_0$, $d_{l_0}(t) \not\equiv 0$ and $t_i(\theta) \equiv t_i(\theta; t_0, \Xi)$. The $t_i(\theta)$ can be expanded into a convergent Puiseux series of θ in $[0, \theta_0]$, with a modification of θ_0 if necessary. Put

$$\begin{aligned} &\mathcal{R}_0(\Xi(\theta); p) = \{t_0 + t_i(\theta);\ 1 \le i \le n_0\}, \\ &\widetilde{\mathcal{R}}_0(\Xi(\theta); p) = \{(t_0 + \operatorname{Re} t_i(\theta))_+;\ 1 \le i \le n_0\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\mathcal{R}_0(\Xi(\theta)) \supset \widetilde{\mathcal{R}}_0(\Xi(\theta); p) \quad (\ \theta \in (0, \theta_0]), \\ &\min_{s \in \mathcal{R}_0(\Xi(\theta))} |t_0 + T(\theta) - s| \le \min_{s \in \widetilde{\mathcal{R}}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \\ &\quad \le \min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \quad (\ \theta \in (0, \theta_0]). \end{aligned} \tag{3.46}$$

(3.41) implies that $D_3(t, \xi) = 0$ if $D_3(t, \xi)\hat{a}_2(t, \xi) = 0$.

(II) The case where $D_3(t, \Xi(\theta)) \equiv 0$ and $\hat{a}_2(t, \Xi(\theta)) \not\equiv 0$ in (t, θ).

Similarly, we can write

$$\hat{a}_2(t_0+t, \Xi(\theta)) = \theta^{l_0} d(t,\theta) \prod_{i=1}^{n_0} (t - t_i(\theta)), \quad d(t,\theta) \neq 0$$
$$\text{for } (t, \theta) \in [-\delta_0, \delta_0] \times [0, \theta_0],$$

where $t_i(\theta) \equiv t_i(\theta; t_0, \Xi)$ is expanded into a convergent Puiseux series of θ in $[0, \theta_0]$, with modifications of θ_0 and δ_0 if necessary. Since $D_2(t, \xi) = 9\hat{a}_2(t, \xi)^2$, we have also

$$\mathcal{R}_0(\Xi(\theta)) \supset \{(t_0 + \operatorname{Re} t_i(\theta))_+;\ 1 \le i \le n_0\}\, (\equiv \widetilde{\mathcal{R}}_0(\Xi(\theta); p)) \quad (\ \theta \in (0, \theta_0]).$$

Putting $\mathcal{R}_0(\Xi(\theta); p) = \{t_0 + t_i(\theta);\ 1 \le i \le n_0\}$, we have (3.46).

(III) The case where $\hat{a}_2(t, \Xi(\theta)) \equiv 0$ in (t, θ).

By (3.11) we have $\hat{p}(t, \tau, \Xi(\theta)) = \tau^3$ and put

$$\mathcal{R}_0(\Xi(\theta); p) = \widetilde{\mathcal{R}}_0(\Xi(\theta); p) = \emptyset \,(\subset \mathcal{R}_0(\Xi(\theta))),$$

$n_0 = 0$ and $l_0 = \infty$.

Now we define

$$\begin{aligned}
&\hat{\mu}\,(\equiv \hat{\mu}(t_0, \xi^0, T, \Xi)) \;= \{\mathrm{Ord}_{\theta\downarrow 0}\, \hat{a}_2(t_0 + T(\theta), \Xi(\theta))\}/2,\\
&\hat{\mu}_0\,(\equiv \hat{\mu}_0(t_0, \xi^0, T, \Xi)) \;= \{\mathrm{Ord}_{\theta\downarrow 0}\, D_3(t_0 + T(\theta), \Xi(\theta))\}/2 - \hat{\mu},\\
&\mu_1\,(\equiv \mu_1(t_0, \xi^0, T, \Xi))\\
&\;= \mathrm{Ord}_{\theta\downarrow 0}\{\min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \alpha(t_0 + T(\theta), \Xi(\theta))\},\\
&\mu_2\,(\equiv \mu_2(t_0, \xi^0, T, \Xi))\\
&\;= \mathrm{Ord}_{\theta\downarrow 0}\{\min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \beta(t_0 + T(\theta), \Xi(\theta))\},\\
&\mu_3\,(\equiv \mu_3(t_0, \xi^0, T, \Xi))\\
&\;= \mathrm{Ord}_{\theta\downarrow 0}\{\min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s|^2 \hat{c}_1(t_0 + T(\theta), \Xi(\theta))\},\\
&\delta\,(\equiv \delta(t_0, \xi^0, T, \Xi)) \;= \mathrm{Ord}_{\theta\downarrow 0}\{\min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s|\}.
\end{aligned}$$

where

$$\begin{aligned}
&\alpha(t, \xi) = \hat{b}_1(t, \xi) + i\partial_t a_1(t, \xi),\\
&\hat{c}_1(t, \xi) = sub^2\, \sigma(P)(t, -a_1(t, \xi)/3, \xi)
\end{aligned}$$

and $\beta(t, \xi)$ is defined by (3.36) with $k = k_0$, and $\hat{\mu} = \hat{\mu}_0 = \hat{\mu}_0 - \hat{\mu} = \infty$ and $\delta = 0$ in the case (III). Here for $f \in C([0, \theta_0])$ $\mathrm{Ord}_{\theta\downarrow 0} f(\theta) = \nu$ $(\in \mathbf{R})$ means that there is $c \in \mathbf{C} \setminus \{0\}$ satisfying $f(\theta) = c\theta^\nu(1 + o(1))$ as $\theta \downarrow 0$. We write $\mathrm{Ord}_{\theta\downarrow 0} f(\theta) = \infty$ if $f(\theta) = O(\theta^N)$ as $\theta \downarrow 0$ for any $N \in \mathbf{Z}_+$. Note that

$$(\partial_\tau sub\ \sigma(P))(t, A(t, \xi) - a_1(t, \xi)/3, \xi) = 2b_0(t, \xi) A(t, \xi) + \alpha(t, \xi). \tag{3.47}$$

It follows from (3.41) and (3.42) that

$$\begin{aligned}
&\hat{\mu}_0 \geq 2\hat{\mu},\\
&\hat{\mu}_0 = \mathrm{Ord}_{\theta\downarrow 0}\, h_2(t_0 + T(\theta), A(t_0 + T(\theta), \Xi(\theta)), \Xi(\theta); \hat{p})^{1/2}.
\end{aligned}$$

Proposition **3.6.** *If*

$$\min\{\mu_1, \mu_3\} < \hat{\mu} \quad \text{or} \quad \mu_2 < \hat{\mu}_0, \tag{3.48}$$

then "the Cauchy problem (CP) *is not C^∞ well-posed" or "*(CP) *does not have finite propagation property."*

Remark. When one replaces $\mathcal{R}_0(\Xi(\theta);p)$ by $\widetilde{\mathcal{R}}_0(\Xi(\theta);p)$ in the definitions of μ_k ($k=1,2,3$) and δ, one can show that the proposition is valid, using (3.46). It follows from (2.5) and (2.26) that whether (3.48) holds or not does not depend on the order of the product in (2.4), although the μ_k are defined under the factorization (3.45). Indeed, if $1\le k\le r$, $m(k)=3$ and $a(t,\tau,\xi)$ is a polynomial of τ satisfying $a(t,\tau,\xi)=O(h_{m-1}(t,\tau,\xi)^{1/2})$ for $(t,\tau,\xi)\in[0,\delta_1]\times I_k\times(\overline{\Gamma}\cap S^{n-1})$, then there are $b_\mu(t,\xi)$ ($1\le\mu\le 3$) and $C>0$ such that

$$\begin{aligned} a(t,\tau,\xi) &= \sum_{\mu=1}^{3} b_\mu(t,\xi)p_\mu^k(t,\tau,\xi), \\ |b_\mu(t,\xi)| &\le C \quad (1\le\mu\le 3). \end{aligned}$$

So we have

$$|\partial_\tau a(t,\tau,\xi)|\le C\sum_{\mu=1}^{3}|\partial_\tau p_\mu^k(t,\tau,\xi)|\le C' h_1(t,\tau,\xi;p^k)^{1/2}.$$

Corollary **3.7.** *Assume that the the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* $(t_0,\xi^0)\in[0,\delta_1/2]\times(\Gamma\cap S^{n-1})$. *Then we have*

$$\begin{aligned} &\hat\mu_0(t_0,\xi^0,T,\Xi)\le\mu_2(t_0,\xi^0,T,\Xi), \\ &\hat\mu(t_0,\xi^0,T,\Xi)\le\mu_k(t_0,\xi^0,T,\Xi) \quad (k=1,3) \end{aligned}$$

if $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T,Ξ).

Remark. The corollary does not depend on the order of the product in (2.4).

In the rest of this subsection we shall prove Proposition 3.6, and give several lemmas. Assume that (3.48) is satisfied. Then we have $\delta<\infty$ since $\mu_k\ge\delta$ ($k=1,2$) and $\mu_3\ge 2\delta$. Moreover, we have $\hat\mu_0>0$ and $D_3(t_0,\xi^0)=0$. There is $c_0>0$ such that

$$\min_{s\in\mathcal{R}_0(\Xi(\theta);p)}|t_0+T(\theta)-s|\ge c_0\theta^\delta \quad \text{for } \theta\in[0,\theta_0].$$

In the case (III) we may take $c_0=1$ and $n_0=0$. For $v\in\mathbf{R}$ we put

$$T_v(\theta)=T(\theta)+v\theta^\delta. \tag{3.49}$$

In the cases (I) and (II) we have

$$\hat a_2(t_0+T_v(\theta),\Xi(\theta))=\theta^{2\hat\mu}(d(v)+o(1)) \quad \text{as } \theta\downarrow 0,$$

where $d(v) \not\equiv 0$ is a polynomial of v with real coefficients. It is easy to see that

$$
\begin{aligned}
& d(v) > 0 \quad \text{for } v \in [-c_0/2, c_0/2], \\
& \hat{a}_2(t_0 + T_v(\theta), \Xi(\theta))^{1/2} = \theta^{\hat{\mu}}(\sqrt{d(v)} + o(1)) \\
& \qquad\qquad \text{uniformly in } v \in [-c_0/2, c_0/2] \text{ as } \theta \downarrow 0.
\end{aligned}
$$

Write

$$
\begin{aligned}
& \alpha(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_1 - \delta}(d_1(v) + o(1)) \quad \text{as } \theta \downarrow 0 \\
& \qquad\qquad \text{if } \alpha(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta), \\
& \beta(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_2 - \delta}(d_2(v) + o(1)) \quad \text{as } \theta \downarrow 0 \\
& \qquad\qquad \text{if } \beta(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta), \\
& \hat{c}_1(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_3 - 2\delta}(d_3(v) + o(1)) \quad \text{as } \theta \downarrow 0 \\
& \qquad\qquad \text{if } \hat{c}_1(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),
\end{aligned}
$$

where $\tilde{\mu}_k \in \mathbf{Q}$ and the $d_k(v)$ ($\not\equiv 0$) are polynomials of v. Here, for instance, we put $\tilde{\mu}_2 = \infty$ if $\beta(t, \Xi(\theta)) \equiv 0$ in (t, θ). We note that $\tilde{\mu}_l \le \mu_l$ ($1 \le l \le 3$). It is easy to see that

$$
\{\operatorname{Ord}_{\theta\downarrow 0} D_3(t_0 + T_v(\theta), \Xi(\theta))\}/2 - \hat{\mu} = \hat{\mu}_0 \quad \text{for } v \in [-c_0/2, c_0/2]
$$

in the case (I). We also write

$$
\begin{aligned}
& \hat{a}_3(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\mu_4}(d_4(v) + o(1)) \quad \text{as } \theta \downarrow 0 \\
& \qquad\qquad \text{if } \hat{a}_3(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),
\end{aligned}
$$

where $d_4(v)$ ($\not\equiv 0$) is a polynomial of v with real coefficients. Therefore, there are $v_0 \in (c_0/4, c_0/2)$ and $s_0 > 0$ such that $I_0 \equiv [v_0 - s_0, v_0 + s_0] \subset [c_0/4, c_0/2]$ and

$$
\begin{aligned}
& d_1(v) \neq 0 \quad \text{if } \alpha(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta), \\
& d_2(v) \neq 0 \quad \text{if } \beta(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta), \\
& d_3(v) \neq 0 \quad \text{if } \hat{c}_1(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta), \\
& d_4(v) \neq 0 \quad \text{if } \hat{a}_3(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta)
\end{aligned}
$$

for $v \in I_0$. In particular, we have

$$
\nu(t_0 + T_v(\theta), \Xi(\theta)) = \begin{cases} 1 & \text{if } d_4(v) > 0 \text{ or } \hat{a}_3(t, \Xi(\theta)) \equiv 0 \text{ in } (t, \theta), \\ -1 & \text{if } d_4(v) < 0 \end{cases}
$$

for $v \in I_0$, where

$$\nu(t,\xi) = \begin{cases} 1 & \text{if } \hat{a}_3(t,\xi) \geq 0, \\ -1 & \text{if } \hat{a}_3(t,\xi) < 0. \end{cases}$$

We replace $T(\theta)$ by $T_{v_0}(\theta)$. Then we can assume that $I_0 = [-s_0, s_0]$, $\mu_l = \tilde{\mu}_l$ ($1 \leq l \leq 3$) and $\min\{\mu_1, \mu_3\} < \hat{\mu}$ or $\mu_2 < \hat{\mu}_0$. Let κ and δ' be positive rational constants satisfying $\delta'\kappa < 1$. Moreover, we assume that $\delta' \in (0,1)$ and $1 - \delta'\kappa < \delta\kappa/2$ (see (3.61) below). We make an asymptotic change of variables:

$$t = t(s;\rho) \equiv t_0 + T(\rho^{-\kappa}) + \rho^{-\delta\kappa} s, \quad x = x(y;\rho) \equiv \rho^{\delta'\kappa - 1} y. \tag{3.50}$$

Put

$$P_\rho(s,\sigma,\eta) = P(t(s;\rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta). \tag{3.51}$$

Let K be a compact neighborhood of $(t_0, 0)$ in $\mathbf{R} \times \mathbf{R}^n$, and put

$$V = \{(s,y,\rho^{-1}) \in [-s_0,s_0] \times \mathbf{R}^n \times (0,\rho_0^{-1}];\ |y| \leq 1\},$$

where $\rho_0 > 0$. We choose ρ_0 so that

$$\begin{aligned} &\{(t(s;\rho), x(y;\rho));\ s \in [-s_0,s_0] \text{ and } |y| \leq 1\} \\ &\subset \{(t,x) \in K;\ t \in [0,\delta_1]\} \quad \text{for } \rho \geq \rho_0. \end{aligned} \tag{3.52}$$

Lemma **3.8.** *Let $\psi \in C^\infty(\mathbf{R})$, and let $q(s,\sigma)$ be a polynomial of σ of degree 3. Then we have*

$$\begin{aligned} &e^{-i\psi(s)} q(s, \rho^{\delta\kappa} D_s)(e^{i\psi(s)} u(s)) \\ &= [q(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma)) - \frac{i}{2} q^{(2)}(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma))\rho^{2\delta\kappa}\partial_s^2\psi(s) \\ &\quad - \frac{1}{6} q^{(3)}(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma))\rho^{3\delta\kappa}\partial_s^3\psi(s)]_{\sigma = D_s} u(s) \end{aligned}$$

for $u(s) \in C^\infty(\mathbf{R})$, where $q^{(k)}(s,\sigma) = \partial_\sigma^k q(s,\sigma)$. Here $a(s,\sigma)|_{\sigma=D_s} = a(s,D_s)$ for a symbol $a(s,\sigma)$.

Proof. If $q(s,\sigma) = \sigma$, σ^2 or σ^3, then the lemma can be easily proved. This proves the lemma. □

Let $\varepsilon = \pm 1$, and let ν_0 and γ_0 be positive constants. Put

$$\varphi(s;\rho) = \sum_{k=0}^{\hat{l}} \rho^{-k\gamma_0} \varphi_k(s;\rho), \tag{3.53}$$

$$\Phi(s,y;\rho)=\rho^{1-\delta\kappa}\int_0^s\{A(t(u;\rho),\Xi(\rho^{-\kappa}))-a_1(t(u;\rho),\Xi(\rho^{-\kappa}))/3\}\,du$$
$$+\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa}),$$
$$E(s,y;\rho,\varepsilon,\nu_0,\varphi)=\exp[i\varepsilon\Phi(s,y;\rho)+i\rho^{\nu_0}\varphi(s;\rho)], \tag{3.54}$$

where $\varphi_k(s;\rho)\in C^\infty([-s_0,s_0])$ for $\rho\geq\rho_0$, $\varphi_k(s;\rho)$ satisfy $|\partial_s^l\varphi_k(s;\rho)|\leq C_l$ for $l\in\mathbf{Z}_+$ and $(s,\rho^{-1})\in[-s_0,s_0]\times(0,\rho_0^{-1}]$, $\hat{l}=0$ or 1, and $A(t,\xi)\,(\equiv A^{k_0}(t,\xi))$ is defined by (3.9) with $k=k_0$. By Lemma 3.8 we have

$$\begin{aligned}
&\widetilde{P}(s,D_s;\rho,E)u(s)\\
&\equiv E(s,y;\rho,\varepsilon,\nu_0,\varphi)^{-1}P_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,\nu_0,\varphi)u(s))\\
&=E(s,0;\rho,\varepsilon,\nu_0,\varphi)^{-1}P(t(s;\rho),\rho^{\delta\kappa}D_s,\varepsilon\rho\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,\nu_0,\varphi)u(s))\\
&=[P(t(s;\rho),\varepsilon\rho\widetilde{A}(s;\rho)+\rho^{\delta\kappa+\nu_0}\partial_s\varphi+\rho^{\delta\kappa}\sigma,\varepsilon\rho\Xi(\rho^{-\kappa}))\\
&\qquad-\frac{i}{2}P^{(2)}(t(s;\rho),\varepsilon\rho\widetilde{A}+\rho^{\delta\kappa+\nu_0}\partial_s\varphi+\rho^{\delta\kappa}\sigma,\varepsilon\rho\Xi(\rho^{-\kappa}))\\
&\qquad\quad\times\rho^{2\delta\kappa}(\varepsilon\rho^{1-\delta\kappa}\partial_s\widetilde{A}+\rho^{\nu_0}\partial_s^2\varphi)\\
&\qquad-\rho^{3\delta\kappa}(\varepsilon\rho^{1-\delta\kappa}\partial_s^2\widetilde{A}+\rho^{\nu_0}\partial_s^3\varphi)]_{\sigma=D_s}u(s),
\end{aligned}$$

where
$$\widetilde{A}\equiv\widetilde{A}(s;\rho)=A(t(s;\rho),\Xi(\rho^{-\kappa}))-a_1(t(s;\rho),\Xi(\rho^{-\kappa}))/3$$
and $\varphi=\varphi(s;\rho)$.

LEMMA **3.9.** *Let $\mu\in\mathbf{Z}_+$, and let $a(s,\theta)\in C^\infty([-s_0,s_0]\times[0,\theta_0])$ satisfy*

$$a(s,\theta)=O(\theta^\mu)\quad\textit{uniformly in } s\in[-s_0,s_0])\textit{ as }\theta\downarrow 0.$$

Namely, there is $C>0$ such that

$$|\theta^{-\mu}a(s,\theta)|\leq C\quad\textit{if }(s,\theta)\in[-s_0,s_0]\times(0,\theta_0].$$

Then, for any $l\in\mathbf{Z}_+$

$$\partial_s^l a(s,\theta)=O(\theta^\mu)\quad\textit{uniformly in } s\in[-s_0,s_0]\textit{ as }\theta\downarrow 0.$$

REMARK. For instance, for $\hat{a}_2(t,\xi)$ there is $L\in\mathbf{N}$ such that

$$a(s,\theta)\equiv\hat{a}_2(t(s;\theta^{-L/\kappa}),\Xi(\theta^L))\in C^\infty([-s_0,s_0]\times[0,\theta_0^{1/L}]).$$

Then, we can apply the lemma to $a(s,\theta)$, and for any $l\in\mathbf{Z}_+$ we have

$$\partial_s^l\hat{a}_2(t(s;\rho),\Xi(\rho^{-\kappa}))=O(\rho^{-\hat{\mu}\kappa})\quad\text{uniformly in } s\in[-s_0,s_0]\text{ as }\rho\to\infty.$$

Proof. By assumption we have

$$(\partial_\theta^l a)(s,0) \equiv 0 \quad \text{in } s \ (\ 0 \le l \le \mu - 1).$$

Then Taylor's formula yields

$$\partial_s^l a(s,\theta) = \frac{\theta^\mu}{(\mu-1)!}\int_0^1 (1-\tau)^{\mu-1}(\partial_s^l \partial_\theta^\mu a)(s,\tau\theta)\, d\tau.$$

This proves the lemma. $\square$

Recall that

$$\begin{aligned}
&\hat{p}(t,\tau,\xi) = p(t,\tau - a_1(t,\xi)/3,\xi) = \tau^3 - \hat{a}_2(t,\xi)\tau + \hat{a}_3(t,\xi),\\
&P(t,\tau,\xi) = p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi) + r(t,\tau,\xi),\\
&q_0(t,\tau - a_1(t,\xi)/3,\xi) = b_0(t,\xi)\tau^2 + \hat{b}_1(t,\xi)\tau + \hat{b}_2(t,\xi),\\
&\alpha(t,\xi) = \hat{b}_1(t,\xi) + i\partial_t a_1(t,\xi).
\end{aligned}$$

A straightforward calculation yields

$$\begin{aligned}
&\widetilde{P}(s,D_s;\rho,E)u(s)\\
&= [\varepsilon\rho^3 \hat{p}(t(s,\rho),A(s;\rho),\Xi(\rho^{-\kappa})) + 3\varepsilon\rho^{1+2\delta\kappa}A(s,\rho)(\rho^{\nu_0}\partial_s\varphi + \sigma)^2\\
&\quad + \rho^{3\delta\kappa}(\rho^{\nu_0}\partial_s\varphi+\sigma)^3 + \rho^2 q_0(t(s,\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa}))\\
&\quad + \varepsilon\rho^{1+\delta\kappa}q_0^{(1)}(t(s,\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi+\sigma)\\
&\quad + \rho^{2\delta\kappa}b_0(t(s,\rho),\Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi+\sigma)^2\\
&\quad + \varepsilon\rho q_1(t(s,\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa}))\\
&\quad + \rho^{\delta\kappa}q_1^{(1)}(t(s,\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi+\sigma)\\
&\quad + \frac{\varepsilon}{2}\rho^{2\delta\kappa-1}q_1^{(2)}(t(s,\rho),0,\Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi+\sigma)^2\\
&\quad + r(t(s;\rho),\varepsilon\rho\widetilde{A}(s;\rho) + \rho^{\delta\kappa+\nu_0}\partial_s\varphi + \rho^{\delta\kappa}\sigma,\varepsilon\rho\Xi(\rho^{-\kappa}))\\
&\quad - \{3\varepsilon i\rho A(s;\rho) + 3i\rho^{\delta\kappa}(\rho^{\nu_0}\partial_s\varphi+\sigma) + ib_0(t(s,\rho),\Xi(\rho^{-\kappa}))\\
&\qquad + \frac{i}{2}q_1^{(2)}(t(s;\rho),0,\varepsilon\rho\Xi(\rho^{-\kappa})) + \frac{i}{2}r^{(2)}(t(s;\rho),0,\varepsilon\rho\Xi(\rho^{-\kappa}))\}\\
&\qquad \times (\varepsilon\rho^{1+\delta\kappa}\partial_s\widetilde{A}(s;\rho) + \rho^{2\delta\kappa+\nu_0}\partial_s^2\varphi)\\
&\quad - (\varepsilon\rho^{1+2\delta\kappa}\partial_s^2\widetilde{A}(s;\rho) + \rho^{3\delta\kappa+\nu_0}\partial_s^3\varphi)]_{\sigma=D_s}u(s)\\
&= [\rho^{3\delta\kappa+3\nu_0}(\partial_s\varphi)^3\\
&\quad + \rho^2\{q_0(t(s;\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa})) - 3i\rho^{\delta\kappa}A(s;\rho)\partial_s\widetilde{A}(s;\rho)\}\\
&\quad + 3\varepsilon\rho^{1+2\delta\kappa+2\nu_0}A(s;\rho)(\partial_s\varphi)^2
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\rho^{1+\delta\kappa+\nu_0}\{\hat{b}_1(t(s;\rho),\Xi(\rho^{-\kappa}))+\rho^{\delta\kappa}\partial_s a_1(t(s;\rho),\Xi(\rho^{-\kappa}))\}\\
&\quad\times(\partial_s\varphi+\rho^{-\nu_0}D_s)\\
&+\varepsilon\rho\{q_1(t(s;\rho),\widetilde{A}(s;\rho),\Xi(\rho^{-\kappa}))+\rho^{2\delta\kappa}\partial_s^2 a_1(t(s;\rho),\Xi(\rho^{-\kappa}))/3\\
&\qquad+i\rho^{\delta\kappa}b_0(t(s;\rho),\Xi(\rho^{-\kappa})\partial_s a_1(t(s;\rho),\Xi(\rho^{-\kappa}))/3\}\\
&-2\varepsilon\rho^3(A(s;\rho)^3-\hat{a}_3(t(s;\rho),\Xi(\rho^{-\kappa}))/2)\\
&+\rho^{3\delta\kappa+2\nu_0}\{3(\partial_s\varphi)^2D_s-3i(\partial_s\varphi)(\partial_s^2\varphi)\\
&\quad+\rho^{-\nu_0}l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,\varepsilon)\\
&\quad+\rho^{-\delta\kappa}b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)^2+\rho^{-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s)\}\\
&+\varepsilon\rho^{1+2\delta\kappa+\nu_0}\{6A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)D_s\\
&\quad-3iA(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s^2\varphi)\\
&\quad-3i\partial_s A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)+\rho^{-\hat{\mu}\kappa-\nu_0}l_3(s,\rho^{-1};D_s)\\
&\quad+2\rho^{-\delta\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))\\
&\qquad\times(\partial_s\varphi+\rho^{-\nu_0}D_s)\}]u(s),\\
=&\,[\rho^{3\delta\kappa+3\nu_0}(\partial_s\varphi)^3+\rho^{2-\mu_2\kappa+\delta\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(t(s;\rho),\Xi(\rho^{-\kappa})))\\
&+3\varepsilon\rho^{1+2\delta\kappa-\hat{\mu}\kappa+2\nu_0}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi)^2\\
&+\varepsilon\rho^{1+2\delta\kappa-\mu_1\kappa+\nu_0}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi+\rho^{-\nu_0}D_s)\\
&+\varepsilon\rho^{1-\mu_3\kappa+2\delta\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(s;\rho),\Xi(\rho^{-\kappa})))\\
&-2\varepsilon\rho^{3-2\hat{\mu}_0+\hat{\mu}\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(t(s;\rho),\Xi(\rho^{-\kappa}))^3-\hat{a}_3(t(s;\rho),\Xi(\rho^{-\kappa}))/2))\\
&+\rho^{3\delta\kappa+2\nu_0}\{3(\partial_s\varphi)^2D_s-3i(\partial_s\varphi)(\partial_s^2\varphi)\\
&\quad+\rho^{-\nu_0}l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,\varepsilon)+\rho^{-\delta\kappa}b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)^2\\
&\quad+\rho^{-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s)\}\\
&+\varepsilon\rho^{1+2\delta\kappa-\hat{\mu}\kappa+\nu_0}\{6(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi)D_s\\
&\quad-3i(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s^2\varphi)\\
&\quad-3i(\rho^{\hat{\mu}\kappa}\partial_s A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi)+\rho^{-\nu_0}l_3(s,\rho^{-1};D_s)\\
&\quad+2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi+\rho^{-\nu_0}D_s)\}]u(s),
\end{aligned}
$$

where $l_1(s,\theta;X_1,X_2,X_3,X_4,X_5)$ is a polynomial of $\{X_k\}_{1\le k\le 5}$ with coefficients in $C^\infty([-s_0,s_0]\times[0,1])$, $\deg_{X_1} l_1=2$, $\deg_{X_k} l_1=1$ ($k=2,3,5$) and $\deg_{X_4} l_1=3$, $l_2(s,\theta;X_1,X_2)$ is a polynomial of X_1 and X_2 with coefficients in $C^\infty([-s_0,s_0]\times[0,1])$, $\deg_{X_k} l_2=2$ $(k=1,2)$, and $l_3(s,\theta;D_s)$ is a differential operator of order 2 with coefficients in $C^\infty([-s_0,s_0]\times[0,1])$, and $l_3(s,\theta;D_s)=0$ if $\hat{\mu}=\infty$. Here we have used the facts that

$$
\begin{aligned}
&3A(t,\xi)^2=\hat{a}_2(t,\xi),\\
&\beta(\cdot)=q_0(t(s;\rho),A(\cdot)-a_1(\cdot)/3,\Xi(\rho^{-\kappa}))+i\rho^{\delta\kappa}A(\cdot)\partial_s a_1(\cdot)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{i}{2}\rho^{\delta\kappa}\partial_s \hat{a}_2(\cdot) \\
&= q_0(t(s;\rho), A(\cdot) - a_1(\cdot)/3, \Xi(\rho^{-\kappa})) - 3i\rho^{\delta\kappa} A(\cdot)\partial_s(A(\cdot) - a_1(\cdot)/3), \\
&\hat{b}_1(\cdot) + i\rho^{\delta\kappa}\partial_s a_1(\cdot) = \alpha(\cdot), \\
&q_1(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa})) + \rho^{2\delta\kappa}\partial_s^2 a_1(\cdot)/3 + i\rho^{\delta\kappa} b_0(\cdot)\partial_s a_1(\cdot)/3 = \hat{c}_1(\cdot), \\
&q_1(t(s;\rho), \widetilde{A}(\cdot), \Xi(\rho^{-\kappa})) \\
&= q_1(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa})) + q_1^{(1)}(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa}))A(\cdot) \\
&\quad + \frac{1}{2} q_1^{(2)}(t(s;\rho), 0, \Xi(\rho^{-\kappa}))A(\cdot)^2, \\
&\begin{cases} r(t(s;\rho), \varepsilon\rho\widetilde{A}(\cdot), \varepsilon\rho\Xi(\rho^{-\kappa})) = O(1), \\ r^{(1)}(t(s;\rho), \varepsilon\rho\widetilde{A}(\cdot), \varepsilon\rho\Xi(\rho^{-\kappa})) = O(\rho^{-1}), \\ r^{(2)}(t(s;\rho), 0, \varepsilon\rho\Xi(\rho^{-\kappa})) = O(\rho^{-2}) \end{cases} \\
&\qquad \text{uniformly in } s \in [-s_0, s_0] \text{ as } \rho \to \infty,
\end{aligned}
$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. We note that

$$
\begin{aligned}
&A(t,\xi)^3 - \hat{a}_3(t,\xi)/2 = \nu(t,\xi)\{(\hat{a}_2(t,\xi)/3)^{3/2} - |\hat{a}_3(t,\xi)|/2\}, \\
&D_3(t,\xi) = 108\{(\hat{a}_2(t,\xi)/3)^{3/2} - |\hat{a}_3(t,\xi)|/2\}\{(\hat{a}_2(t,\xi)/3)^{3/2} + |\hat{a}_3(t,\xi)|/2\}, \\
&D_3(t,\xi) \le 216|A(t,\xi)^3 - \hat{a}_3(t,\xi)/2|(\hat{a}_2(t,\xi)/3)^{3/2} \le 2D_3(t,\xi).
\end{aligned}
$$

This implies that there is $C > 0$ satisfying

$$
\rho^{2\hat{\mu}_0\kappa - \hat{\mu}\kappa}|A(t(s;\rho), \Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2| \le C
$$

for $(s, \rho^{-1}) \in [-s_0, s_0] \times (0, \rho_0^{-1}]$. We shall prove Proposition 3.6 by dividing into four cases:

Case A is the case where

$$
\min\{\mu_1, \mu_3\} \ge \mu_2/2 \quad \text{and} \quad \mu_2 < 2\hat{\mu}.
$$

Case B is the case where

$$
\min\{\mu_1, \mu_3\} \ge \hat{\mu} \quad \text{and} \quad 2\hat{\mu} \le \mu_2 < \hat{\mu}_0.
$$

Case C is the case where

$$
\mu_1 \le \mu_3, \quad 2\mu_1 < \mu_2 \quad \textit{and} \quad \mu_1 < \hat{\mu}.
$$

Case D is the case where

$$
\mu_3 \le \mu_1, \quad 2\mu_3 < \mu_2 \quad \textit{and} \quad \mu_3 < \hat{\mu}.
$$

Let us first consider Case A. We choose

$$\nu_0 = (2 - \mu_2\kappa - 2\delta\kappa)/3, \quad \kappa = (\delta + \mu_2/2 + 3\mu_4/2)^{-1},$$

where $\mu_4 = \min\{\hat{\mu} - \mu_2/2, 2/3\}$. Then we have

$$\begin{aligned}
&3\delta\kappa + 3\nu_0 = 2 - \mu_2\kappa + \delta\kappa, \quad 1 - \delta\kappa = (\mu_2 + 3\mu_4)\kappa/2, \\
&\nu_0 = \mu_4\kappa\,(>0), \quad \nu_0 \le 2/3, \\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0/2 + (\hat{\mu} - \mu_2/2 - \mu_4)\kappa \ge \nu_0/2, \\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0/2 + (\mu_1 - \mu_2/2)\kappa \ge \nu_0/2, \\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_3\kappa) = 3\nu_0/2 + (\mu_3 - \mu_2/2)\kappa \ge 3\nu_0/2, \\
&3\delta\kappa + 3\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) \\
&\ = 3\nu_0/2 + 2(\hat{\mu}_0 - 2\hat{\mu})\kappa + 3(\hat{\mu} - \mu_2/2 - \mu_4)\kappa \ge 3\nu_0/2.
\end{aligned}$$

So we choose $\varepsilon = 1$, $\hat{l} = 1$ and $\gamma_0 = \nu_0/2$ in (3.53) and (3.54). We note that

$$A(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv A(t(s;\rho), \Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2 \equiv 0$$

and $3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa = -\infty$ when $\hat{\mu} = \infty$. Define $\varphi_0(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi_0(s;\rho) = \int_0^s (-\rho^{\mu_2\kappa - \delta\kappa}\beta(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/3}\, du. \tag{3.55}$$

Note that

$$\rho^{\mu_2\kappa - \delta\kappa}\beta(t(s;\rho), \Xi(\rho^{-\kappa})) = d_2(s) + o(1) \quad \text{as } \rho \to \infty,$$

where $d_2(s) \ne 0$ for $s \in [-s_0, s_0]$. Here we have chozen a branch of $(-\rho^{\mu_2\kappa - \delta\kappa} \times \beta(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/3}$ so that its imaginary part is negative. Then there is $\hat{d} > 0$ such that

$$\operatorname{Im} \varphi_0(s;\rho) \ge \hat{d}|s| \quad \text{for } s \in [-s_0, 0) \text{ and } \rho \ge \rho_0,$$

with a modification of ρ_0 if necessary. Since

$$\begin{aligned}
&(\partial_s\varphi_0(s;\rho) + \rho^{-\nu_0/2}\partial_s\varphi_1(s;\rho))^3 \\
&\ = (\partial_s\varphi_0)^3 + 3\rho^{-\nu_0/2}(\partial_s\varphi_0)^2(\partial_s\varphi_1) + 3\rho^{-\nu_0}(\partial_s\varphi_0)(\partial_s\varphi_1)^2 + \rho^{-3\nu_0/2}(\partial_s\varphi_1)^3,
\end{aligned}$$

$\partial_s\varphi_1(s;\rho)$ is chosen so as to satisfy

$$3(\partial_s\varphi_0)^2(\partial_s\varphi_1) + 3\rho^{-(\hat{\mu} - \mu_2/2 - \mu_4)}(\rho^{\hat{\mu}\kappa}A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi_0)^2$$

$$+ \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi_0) = 0.$$

Noting $\partial_s\varphi_0(s;\rho) = (-d_2(s))^{1/3} + O(1)$ as $\rho \to \infty$, we define

$$\varphi_1(s;\rho) = -\int_0^s [\rho^{-(\hat{\mu}-\mu_2/2-\mu_4)}(\rho^{\hat{\mu}\kappa}A(t(u;\rho),\Xi(\rho^{-\kappa})))$$
$$+ \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho),\Xi(\rho^{-\kappa}))/(3(\partial_s\varphi_0)(u;\rho))]\,du.$$

Putting

$$(3.56)\qquad u(s;\rho^{-1}) \sim \sum_{l=0}^{\infty} \rho^{-l\nu_0}u_l(s;\rho^{-1}),$$

$$(3.57)\qquad u_{-1}(s;\rho^{-1}) \equiv 0, \quad u_0(0;\rho^{-1}) = 1, \quad u_k(0;\rho^{-1}) = 0 \ (\ k \geq 1),$$

we obtain the following transport equations for $u(s;\rho^{-1})$:

$$
\begin{aligned}
(3.58)\qquad & \{(3(\partial_s\varphi(s;\rho))^2 + 6\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi) \\
& \quad + \rho^{-\nu_0/2-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))D_s \\
& + 3(\partial_s\varphi_1(s;\rho))^2\partial_s\varphi_0(s;\rho) + \rho^{-\nu_0/2}(\partial_s\varphi_1)^3 \\
& + 6\rho^{-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi_0)(\partial_s\varphi_1) \\
& + 3\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi_1)^2 \\
& + \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi_1) \\
& + \rho^{-\nu_0/2-(\mu_3-\mu_2/2)\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(s;\rho),\Xi(\rho^{-\kappa}))) \\
& - 2\rho^{-\nu_0/2-2(\hat{\mu}_0-2\hat{\mu})\kappa-3\nu_1} \\
& \quad \times (\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(t(s;\rho),\Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho),\Xi(\rho^{-\kappa}))/2)) \\
& - 3i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)^2 \\
& - 3i\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s^2\varphi) \\
& - 3i\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}\partial_sA(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi) \\
& + 2\rho^{-\delta\kappa-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)\} \\
& \qquad \times u_k(s;\rho^{-1}) \\
& + \{l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,1) \\
& \quad + \rho^{\nu_0-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s) + \rho^{-\nu_0/2-\nu_1}l_3(s,\rho^{-1},D_s) \\
& \quad + 2\rho^{-\delta\kappa-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))D_s\} \\
& \qquad \times u_{k-1}(s;\rho^{-1}) = 0 \quad (\ k = 0,1,2,\cdots),
\end{aligned}
$$

where $\nu_1 = (\hat{\mu} - \mu_2/2 - \mu_4)\kappa$. We can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\cdots}$, inductively, so as to satisfy (3.57) and (3.58). It is easy to see that there are

$C_{l,k} > 0$ $(l, k \in \mathbf{Z}_+)$ satisfying

$$|D_s^l u_k(s;\rho^{-1})| \le C_{l,k} \quad \text{for } l,k \in \mathbf{Z}_+,\ s \in [-s_0, s_0] \text{ and } \rho \in [\rho_0, \infty).$$

Let $\phi(s) \in C_0^\infty(\mathbf{R})$ satisfy

$$\phi(s) = \begin{cases} 1 & \text{if } |s| \le s_0/2, \\ 0 & \text{if } |s| \ge s_0, \end{cases}$$

and put

$$v_N(s,y;\rho^{-1},\varepsilon) = \sum_{k=0}^{N} \rho^{-k\nu_0} u_k(s;\rho^{-1})\phi(s) E(s,y;\rho,\varepsilon,\nu_0,\varphi) \qquad (N \in \mathbf{Z}_+). \tag{3.59}$$

Then we have

$$\begin{aligned} &(\rho^{\delta\kappa} D_s)^l (\rho^{1-\delta'\kappa} D_y)^\alpha P_\rho(s, D_s, D_y) v_N(s,y;\rho^{-1},1) \\ &= \begin{cases} O(\rho^{3\delta\kappa+2\nu_0-\nu_0(N+1)+l+|\alpha|}) \\ \quad \text{unniformly in } \widetilde{\Omega}_{\varepsilon_0,\rho}(0,0) \cap \{|s| \le s_0/2\} \text{ as } \rho \to \infty, \\ O(\rho^{-M}) \\ \quad \text{unniformly in } \widetilde{\Omega}_{\varepsilon_0,\rho}(0,0) \cap \{s_0/2 \le |s| \le s_0\} \text{ as } \rho \to \infty \\ \quad (M \in \mathbf{N}), \end{cases} \end{aligned} \tag{3.60}$$

where

$$\widetilde{\Omega}_{\varepsilon_0,\rho}(0,0) = \{(s,y) \in \mathbf{R}^{n+1};\ s < -\varepsilon_0 \rho^{\delta\kappa - 2 + 2\delta'\kappa} |y|^2\}. \tag{3.61}$$

Here we have taken $\varepsilon = 1$ in Case A. Next cosider Case B. Note that $\hat{\mu} < \infty$ and $\mu_2 < \infty$. We choose

$$\nu_0 = (1 - \delta\kappa + \hat{\mu}\kappa - \mu_2\kappa)/2, \quad \kappa = (\delta - \hat{\mu} + \hat{\mu}_0)^{-1}.$$

Then we have

$$\begin{aligned} &2 - \mu_2\kappa + \delta\kappa = 1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0, \quad 1 - \delta\kappa = (\hat{\mu}_0 - \hat{\mu})\kappa, \\ &\nu_0 = (\hat{\mu}_0 - \mu_2)\kappa/2\ (> 0), \\ &1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (3\delta\kappa + 3\nu_0) = \nu_0 + (\mu_2 - 2\hat{\mu})\kappa \ge \nu_0, \\ &1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0 + (\mu_1 - \hat{\mu})\kappa \ge \nu_0, \\ &1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (1 - \mu_3\kappa + 2\delta\kappa) = 2\nu_0 + (\mu_3 - \hat{\mu})\kappa \ge 2\nu_0, \end{aligned}$$

$$1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) = 2\nu_0.$$

Therefore, we choose $\hat{l} = 0$ in (3.53) and $\varepsilon = \pm 1$ so that the imaginary part of a branch of $(-\varepsilon\tilde{\beta}(s;\rho)/3)^{1/2}$ is negative, where

$$\tilde{\beta}(s;\rho) = (\rho^{\hat{\mu}\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa})))^{-1}(\rho^{\mu_2\kappa-\delta\kappa}\beta(t(s;\rho), \Xi(\rho^{-\kappa})).$$

We define $\varphi(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi(s;\rho) = \int_0^s (-\varepsilon\tilde{\beta}(u;\rho)/3)^{1/2}\, du.$$

Here we have

$$\begin{aligned} &\tilde{\beta}(s;\rho) = (d(s)/3)^{-1/2} d_2(s) + o(1) \quad \text{as } \rho \to \infty, \\ &d(s) > 0 \quad \text{and} \quad d_2(s) \neq 0 \quad \text{for } s \in [-s_0, s_0], \\ &\mathrm{Im}(-\varepsilon\tilde{\beta}(s;\rho)/3)^{1/2} < 0 \quad \text{for } s \in [-s_0, s_0]. \end{aligned}$$

Writing $u(s;\rho^{-1})$ as (3.56), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned} (3.62) \quad &\{6\varepsilon(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)D_s + \rho^{-(\mu_2-2\hat{\mu})\kappa}(\partial_s\varphi)^3 \\ &\quad + \varepsilon\rho^{-\kappa(\mu_1-\hat{\mu})}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))(\partial_s\varphi) - 3\varepsilon i(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s^2\varphi) \\ &\quad - 3\varepsilon i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi) + 2\varepsilon\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi)\}u_k(s;\rho^{-1}) \\ &+ \{\varepsilon\rho^{-\kappa(\mu_1-\hat{\mu})}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))D_s + \varepsilon\rho^{-\kappa(\mu_3-\hat{\mu})}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(\cdot)) \\ &\quad - 2\varepsilon(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2)) \\ &\quad + 3\rho^{-\kappa(\mu_2-2\hat{\mu})}((\partial_s\varphi)^2 D_s - i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2 \\ &\qquad + \rho^{-\nu_0}l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,\varepsilon) \\ &\qquad + \rho^{-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s)) \\ &\quad + 2\varepsilon\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)D_s + \varepsilon l_3(s,\rho^{-1};D_s)\}u_{k-1}(s;\rho^{-1}) = 0 \\ &\qquad\qquad (\, k = 0,1,2,\cdots), \end{aligned}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. Similarly, we can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\cdots}$ so as to satisfy (3.57) and (3.62). We define $v_N(s,y;\rho^{-1},\varepsilon)$ ($N \in \mathbf{Z}_+$) by (3.59). Then we have (3.60), replacing 1 by ε on the left-hand side and $O(\rho^{3\delta\kappa+2\nu_0-\nu_0(N+1)+l+|\alpha|})$ by $O(\rho^{2-\mu_2\kappa+\delta\kappa-\nu_0(N+2)+l+|\alpha|})$ on the right-hand side. Let us consider Case C. Note that $\mu_1 < \infty$ and $\min\{\mu_2 - \mu_1, \hat{\mu}\} > 0$. We choose

$$\nu_0 = (1 - \delta\kappa - \mu_1\kappa)/2, \quad \kappa = (\delta + \mu_1 + \mu_5)^{-1},$$

where

$$\mu_5 = \min\{\mu_2 - 2\mu_1, \hat{\mu} - \mu_1, 1\}\kappa.$$

Then we have

$$\begin{aligned}
&3\delta\kappa + 3\nu_0 = 1 + 2\delta\kappa - \mu_1\kappa + \nu_0, \quad 1 - \delta\kappa = \mu_1\kappa + \mu_5\kappa,\\
&\nu_0 = (1 - \delta\kappa - \mu_1\kappa)/2 = \mu_5\kappa/2\,(> 0), \quad \nu_0 \le 1/2,\\
&3\delta\kappa + 3\nu_0 - (2 + \delta\kappa - \mu_2\kappa) = \nu_0 + (\mu_2 - 2\mu_1 - \mu_5)\kappa \ge \nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0 + (\hat{\mu} - \mu_1 - \mu_5)\kappa \ge \nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_3\kappa) = \nu_0 + (\mu_3 - \mu_1)\kappa \ge \nu_0,\\
&3\delta\kappa + 3\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) = 3\nu_0 + 2(\hat{\mu}_0 - 2\hat{\mu})\kappa + 3(\hat{\mu} - \mu_1 - \mu_5)\kappa \ge 3\nu_0.
\end{aligned}$$

We note that

$$\beta(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv 0 \quad \text{in } s \text{ for } \rho \ge \rho_0 \quad \text{when } \mu_2 = \infty.$$

So we choose $\hat{l} = 0$ in (3.53) and $\varepsilon = \pm 1$ so that the imaginary part of a branch of $(-\varepsilon\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho), \Xi(\rho^{-\kappa})))^{1/2}$ is negative. We define $\varphi(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi(s;\rho) = \int_0^s (-\varepsilon\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/2}\, du. \tag{3.63}$$

Writing $u(s;\rho^{-1})$ as (3.56), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{2(\partial_s\varphi)^2 D_s + \rho^{-(\mu_2-2\mu_1-\mu_5)\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(\cdot))\\
&\quad + 3\varepsilon\rho^{-(\hat{\mu}-\mu_1-\mu_5)\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)^2 + \varepsilon\rho^{-(\mu_3-\mu_1)\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(\cdot))\\
&\quad - 3i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2\}u_k(s;\rho^{-1})\\
&+ \{-2\varepsilon\rho^{-\nu_0-3(\hat{\mu}-\mu_1-\mu_5)\kappa-2(\hat{\mu}_0-2\hat{\mu})\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2))\\
&\quad + l_1(s, \rho^{-1}; \partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, \varepsilon) + \rho^{\nu_0-\delta\kappa-1}l_2(s, \rho^{-1}; \partial_s\varphi, D_s)\\
&\quad + \varepsilon\rho^{-(\hat{\mu}-\mu_1-\mu_5)\kappa}(6(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)D_s - 3i(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s^2\varphi)\\
&\qquad - 3i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi) + \rho^{-\nu_0}l_3(s, \rho^{-1}; D_s)\\
&\qquad + 2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi + \rho^{-\nu_0}D_s))\}\\
&\quad \times u_{k-1}(s;\rho^{-1}) = 0 \quad (\, k = 0, 1, 2, \cdots),
\end{aligned} \tag{3.64}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. Similarly, we can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\cdots}$ so as to satisfy (3.57) and (3.64). We define $v_N(s, y; \rho^{-1}, \varepsilon)$ ($N \in \mathbf{Z}_+$) by (3.59). Then we have (3.60) with an obvious modification. Let us finally consider Case D. Note that $\mu_3 < \infty$. We choose

$$\nu_0 = (1 - \delta\kappa - \mu_3\kappa)/3, \quad \kappa = (\delta + \mu_3 + \mu_6)^{-1},$$

where

$$\mu_6 = \min\{\mu_2/2 - \mu_3, 3(\hat{\mu} - \mu_3)/4, 3(\mu_1 - \mu_3)/2, 1\}.$$

Then we have

$$\begin{aligned}
&3\delta\kappa + 3\nu_0 = 1 + 2\delta\kappa - \mu_3\kappa,\\
&1 - \delta\kappa = (\mu_3 + \mu_6)\kappa = \mu_3\kappa + 3\nu_0,\\
&\nu_0 = \mu_6\kappa/3\,(>0), \quad \nu_0 \le 1/3,\\
&3\delta\kappa + 3\nu_0 - (2 - \mu_2\kappa + \delta\kappa) = 3\nu_0 + 2(\mu_2/2 - \mu_3 - \mu_6)\kappa \ge 3\nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0 + (\hat{\mu} - \mu_3 - \mu_6)\kappa > \nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0 + (\mu_1 - \mu_3 - 2\mu_6/3)\kappa \ge \nu_0,\\
&3\delta\kappa + 3\nu_0 - (3 + \hat{\mu}\kappa - 2\hat{\mu}_0\kappa) = 6\nu_0 + (2(\hat{\mu}_0 - 2\hat{\mu}) + 3(\hat{\mu} - \mu_3) - 4\mu_6)\kappa\\
&\quad \ge 6\nu_0.
\end{aligned}$$

We choose $\varepsilon = 1$ and $\hat{l} = 0$ in (3.53) and (3.54). Define $\varphi(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi(s;\rho) = \int_0^s [-(\rho^{\mu_3\kappa - 2\delta\kappa}\hat{c}_1(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/3}\,du. \tag{3.65}$$

Here we have chosen a branch of $[-(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/3}$ so that its imaginary part is negative. Writing $u(s;\rho^{-1})$ as (3.56), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{3(\partial_s\varphi)^2 D_s - 3i(\partial_s\varphi)(\partial_s^2\varphi) + 3\rho^{-(\hat{\mu}-\mu_3-\mu_6)\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)^2\\
&\quad + \rho^{-(\mu_1-\mu_3-2\mu_6/3)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))(\partial_s\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2\}u_k(s;\rho^{-1})\\
&+ \{\rho^{-(\hat{\mu}-\mu_3-\mu_6)\kappa}(6(\rho^{\hat{\mu}\kappa}A(\cdot)(\partial_s\varphi)D_s - 3i(\rho^{\hat{\mu}\kappa}A(\cdot)(\partial_s^2\varphi) - 3i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi)\\
&\quad + 2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi + \rho^{-\nu_0}D_s) + \rho^{-\nu_0}l_3(s,\rho^{-1};D_s))\\
&\quad + \rho^{-\nu_0-(\mu_2-2\mu_3-2\mu_6)\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(\cdot))\\
&\quad - 2\rho^{-4\nu_0-(2(\hat{\mu}_0-2\hat{\mu})+3(\hat{\mu}-\mu_3)-4\mu_6)\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2))\\
&\quad + l_1(s,\rho^{-1};\partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, 1)\\
&\quad + \rho^{\nu_0-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi, D_s)\}u_{k-1}(s;\rho^{-1}) = 0 \quad (\,k = 0, 1, 2, \cdots).
\end{aligned}$$

Here $l_3(s;\theta, D_s) = 0$ if $\hat{\mu} = \infty$, $\alpha(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv 0$ in s for $\rho \ge \rho_0$ if $\mu_1 = \infty$, and $\beta(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv 0$ in s for $\rho \ge \rho_0$ if $\mu_2 = \infty$. Similarly, we can construct $\{v_N(s, y;\rho^{-1}, 1)\}_{N\in\mathbf{Z}_+}$ satisfying (3.59) with $\varepsilon = 1$ and (3.60).

The condition (3.48) in Proposition 3.6 is satisfied if and only if at least one of Case A to Case D occur. Indeed, first assume that $\min\{\mu_1, \mu_3\} < \hat{\mu}$. If $\mu_1 \le \mu_3$ and $2\mu_1 < \mu_2$, then $\mu_1 < \hat{\mu}$ and Case C occurs. If $\mu_1 \le \mu_3$ and

$\mu_2 \le 2\mu_1$, then $\mu_1 < \hat{\mu}$ and Case A occurs. If $\mu_3 < \mu_1$ and $2\mu_3 < \mu_2$, then $\mu_3 < \hat{\mu}$ and Case D occurs. If $\mu_3 < \mu_1$ and $\mu_2 \le 2\mu_3$, then $\mu_3 < \hat{\mu}$ and Case A occurs. Next assume that $\min\{\mu_1, \mu_3\} \ge \hat{\mu}$ and $\mu_2 < \hat{\mu}_0$. If $\mu_2 < 2\hat{\mu}\,(\le \hat{\mu}_0)$, then Case A occurs. If $2\hat{\mu} \le \mu_2\,(< \hat{\mu}_0)$, then Case B occurs. This proves the "only if" part. The converse is obvious.

Now we won't omit "k_0", *i.e.*, $P(t, D_t, D_x)$ denotes the differential operator in §1.

LEMMA **3.10.** *Assume that the Cauchy problem* (CP) *is C^∞ well-posed and has finite propagation property. Let K be a compact neighborhood of $(t_0, 0)$ in $\mathbf{R} \times \mathbf{R}^n$, and let ρ_0 be a positive constant satisfying* (3.52)*. Then for any $p \in \mathbf{Z}_+$ there are $C > 0$ and $q \in \mathbf{Z}_+$ such that*

$$(3.66)\quad |v|_{p,\widetilde{\Omega}_{\varepsilon_0,\rho}(0,0)} \le C\rho^{q\delta\kappa} |P_\rho(s, D_s, D_y)v|_{q,\widetilde{\Omega}_{\varepsilon_0,\rho}(0,0)} \quad \text{for } \rho \ge \rho_0 \text{ and } v(s,y) \in C^\infty(\mathbf{R}^{n+1}) \text{ with } \operatorname{supp} v \subset \{(s,y);\ t(s;\rho) \ge 0\},$$

where $\widetilde{\Omega}_{\varepsilon_0,\rho}(0,0)$ is defined by (3.61)*, ε_0 is a positive constant defined as* Lemma 3.5*, and $P_\rho(s,\sigma,\eta)$ is defined by* (3.51)*.*

PROOF. Let $v(s,y) \in C^\infty(\mathbf{R}^{n+1})$ satisfy $\operatorname{supp} v \subset W$, where $W = [-s_0, s_0] \times \{y \in \mathbf{R}^n;\ |y| \le 1\}$. Put

$$u_\rho(t,x) = v(\rho^{\delta\kappa}(t - t_0 - T(\rho^{-\kappa})), \rho^{-\delta'\kappa+1}x).$$

Then we have

$$P(t, D_t, D_x)u_\rho(t,x)|_{t=t(s;\rho),\, x=x(y;\rho)} = P_\rho(s, D_s, D_y)v(s,y).$$

It is obvious that

$$(s,y) \in \widetilde{\Omega}_{\varepsilon_0,\rho}(0,0) \Leftrightarrow (t(s;\rho), x(y;\rho)) \in \Omega_{\varepsilon_0}(t_0 + T(\rho^{-\kappa}), 0).$$

Therefore, Lemma 3.5 proves the lemma. □

We factorized $P(t,\tau,\xi)$ as (3.45). Then we have

$$\begin{aligned} &P_\rho(s, D_s, D_y)(\exp[i\varepsilon\rho^{\delta'\kappa} y \cdot \Xi(\rho^{-\kappa})]u(s)) \\ &= \exp[i\varepsilon\rho^{\delta'\kappa} y \cdot \Xi(\rho^{-\kappa})]\{P_\rho^1(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa})) \cdots P_\rho^{k_0-1}(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa})) \\ &\quad \times P_\rho^{k_0+1}(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa})) \cdots P_\rho^{r}(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa})) \\ &\quad \times P_\rho^{k_0}(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa})) + R_\rho(s, D_s, \varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))\}u(s), \end{aligned}$$

where $R_\rho(s,\sigma,\eta) = R(t(s;\rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta)$. For $P_\rho^{k_0}(s, D_s, D_y)$ we constructed asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N \in \mathbf{Z}_+}$ satisfying (3.59) and

(3.60) with an obvious modification when at least one of Case A, Case C and Case D occurs. Here we should choose ε appropriately. In Case B we constructed asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N\in\mathbf{Z}_+}$ satisfying (3.59) and (3.60) with $3\delta\kappa+2\nu_0$ in the exponent replaced by $2-\mu_2\kappa+\delta\kappa$. In (3.60) we replace 1 by ε. Note that

$$\begin{aligned}
&P_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,\nu_0,\varphi)u(s))\\
&\quad=\exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]P_\rho(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,\nu_0,\varphi)u(s)),\\
&R_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,\nu_0,\varphi)u(s))\\
&\quad=\exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]R_\rho(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,\nu_0,\varphi)u(s))\\
&\quad=O(\rho^{-M})\quad\text{uniformly in } [-s_0,0] \text{ as } \rho\to\infty\ (\ M\in\mathbf{N})
\end{aligned}$$

for $u(s)\in C^\infty([-s_0,s_0])$. Therefore, Lemma 3.10 proves Proposition 3.6, since the asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N\in\mathbf{Z}_+}$ violate (3.66).

LEMMA **3.11.** *Assume that $1\le k_0\le r$ and $m(k_0)=3$, and that the Cauchy problem* (CP) *is C^∞ well-posed and has finite propagation property. Let $(t_0,\xi^0)\in[0,\delta_1/2]\times(\Gamma\cap S^{n-1})$, and let $T(\theta)$ and $\Xi(\theta)$ satisfy the condition (T,Ξ).*
(i) *We have*

$$\begin{aligned}
(3.67)\qquad &\mathrm{Ord}_{\theta\downarrow 0}\min_{s\in\mathcal{R}_0(\Xi(\theta);p^{k_0})}|t_0+T(\theta)-s|\\
&\quad\times\, sub\ \sigma(P)(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))\\
&\ge\mathrm{Ord}_{\theta\downarrow 0}\,h_{m-1}(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))^{1/2},\\
(3.68)\qquad &\mathrm{Ord}_{\theta\downarrow 0}\min_{s\in\mathcal{R}_0(\Xi(\theta);p^{k_0})}|t_0+T(\theta)-s|\\
&\quad\times(\partial_\tau sub\ \sigma(P))(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))\\
&\ge\mathrm{Ord}_{\theta\downarrow 0}\,\hat{a}_2^{k_0}(\cdot)^{1/2}\\
&\quad(=\mathrm{Ord}_{\theta\downarrow 0}\,h_{m-2}(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))^{1/2}),
\end{aligned}$$

where $A^{k_0}(t,\xi)$is defined by (3.9) *with $k=k_0$ and $(\cdot)=(t_0+T(\theta),\Xi(\theta))$.*
(ii) *Assume that $\tau_0\in\mathbf{R}$ and $(\partial_\tau^l p^{k_0})(t_0,\tau_0,\xi^0)=0$ $(l=0,1,2)$, and put $z^0=(t_0,\tau_0,\xi^0)$. Then we have*

$$\begin{aligned}
(3.69)\qquad &\mathrm{Ord}_{\theta\downarrow 0}\min_{s\in\mathcal{R}_0(\Xi(\theta);p^{k_0})}|t_0+T(\theta)-s|^2\\
&\quad\times Q(t_0+T(\theta),-a_1(\cdot;z^0)/3,\Xi(\theta);z^0)\\
&\ge\mathrm{Ord}_{\theta\downarrow 0}\,h_{m-2}(t_0+T(\theta),-a_1(\cdot;z^0)/3,\Xi(\theta))^{1/2},
\end{aligned}$$

where $(\cdot;z^0)=(t_0+T(\theta),\Xi(\theta);z^0)$.

Remark. (i) On the assumption that the factorization (2.3) is given near $t = 0$, the lemma is stated. Therefore, if for $t_0 \in [0, \infty)$ the factorization of $p(t, \tau, \xi)$ is given in a neighborhood I of t_0, the lemma is valid with $[0, \delta_1/2]$ replaced by a compact sub-interval of $\overset{\circ}{I}$. (ii) We note that $p(t, \tau, \xi; z^0) = p^{k_0}(t, \tau, \xi)$ and $a_1(t, \xi; z^0) = a_1^{k_0}(t, \xi)$ in the assertion (ii).

Proof. From (2.5) it follows that

$$
\begin{aligned}
(3.70) \qquad & sub\ \sigma(P)(\cdot) = sub\ \sigma(P^{k_0})(\cdot)\Pi_{\{k_0\}}(\cdot) \\
& \quad + \sum_{1 \le k \le r,\, k \ne k_0} sub\ \sigma(P^k)(\cdot) p^{k_0}(\cdot)\Pi_{\{k_0,k\}}(\cdot) + O(h_{m-1}(\cdot)^{1/2}).
\end{aligned}
$$

where $(\cdot) = (t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta))$. On the other hand, by (1.1) we have

$$
\begin{aligned}
(3.71) \qquad & h_{m-1}(t, \tau, \xi) = h_2(t, \tau, \xi; p^{k_0})\Pi_{\{k_0\}}(t, \tau, \xi)^2 \\
& \qquad\qquad + h_{m-4}(t, \tau, \xi; p/p^{k_0})p^{k_0}(t, \tau, \xi)^2, \\
(3.72) \qquad & \mathrm{Ord}_{\theta\downarrow 0}\, p^k(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta)) = 0 \quad \text{if } k \ne k_0,
\end{aligned}
$$

where $h_l(t, \tau, \xi; p) = 0$ if $l < 0$, and $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. Corollary 3.7, (3.70) and (3.71) prove (3.67), since

$$
\begin{aligned}
& \mathrm{Ord}_{\theta\downarrow 0}\, p^{k_0}(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta))^2 \\
& > \mathrm{Ord}_{\theta\downarrow 0}\, h_2(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta); p^{k_0})
\end{aligned}
$$

if $0 < \mathrm{Ord}_{\theta\downarrow 0}\, p^{k_0}(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta)) < \infty$, where $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. It follows from (2.5) that

$$
\begin{aligned}
(3.73) \quad & \partial_\tau sub\ \sigma(P)(t, \tau, \xi) = \partial_\tau sub\ \sigma(P^{k_0})(t, \tau, \xi) \cdot \Pi_{\{k_0\}}(t, \tau, \xi) \\
& + sub\ \sigma(P^{k_0})(t, \tau, \xi)\partial_\tau \Pi_{\{k_0\}}(t, \tau, \xi) \\
& + \sum_{1 \le k \le r,\, k \ne k_0} \{\partial_\tau p^{k_0}(t, \tau, \xi) \cdot sub\ \sigma(P^k)(t, \tau, \xi)\Pi_{\{k_0,k\}}(t, \tau, \xi) \\
& \qquad\qquad + p^{k_0}(t, \tau, \xi)\partial_\tau(sub\ \sigma(t, \tau, \xi)\Pi_{\{k_0,k\}}(t, \tau, \xi))\} \\
& - \frac{i}{2} \sum_{1 \le k \le r,\, k \ne k_0} \{\partial_\tau \{p^k, p^{k_0}\}(t, \tau, \xi) \cdot \Pi_{\{k_0,k\}}(t, \tau, \xi) \\
& \qquad\qquad + \{p^k, p^{k_0}\}(t, \tau, \xi)\partial_\tau \Pi_{\{k_0,k\}}(t, \tau, \xi)\} \\
& - \frac{i}{2} \sum_{1 \le k < l \le r,\, k,l \ne k_0} \{\partial_\tau p^{k_0}(t, \tau, \xi) \cdot \{p^k, p^l\}(t, \tau, \xi)\Pi_{\{k_0,k,l\}}(t, \tau, \xi) \\
& \qquad\qquad + p^{k_0}(t, \tau, \xi)\partial_\tau(\{p^k, p^l\}(t, \tau, \xi)\Pi_{\{k_0,k,l\}}(t, \tau, \xi))\}.
\end{aligned}
$$

Corollary 3.7, (3.72) and (3.73) prove (3.68), since

$$\begin{aligned}&|(\partial_t^\mu \partial_\tau^\nu p^{k_0})(t,\tau,\xi)| \le C h_{3-l}(t,\tau,\xi;p^{k_0})^{1/2} \quad \text{if } l=1,2 \text{ and } \mu+\nu=l,\\ &h_{m-2}(t,\tau,\xi) = h_1(t,\tau,\xi;p^{k_0})\Pi_{\{k_0\}}(t,\tau,\xi)^2\\ &\quad + h_2(t,\tau,\xi;p^{k_0})h_{m-4}(t,\tau,\xi;p/p^{k_0}) + h_{m-5}(t,\tau,\xi;p/p^{k_0})p^{k_0}(t,\tau,\xi)^2.\end{aligned}$$

Next let us prove the assertion (ii). Corollary 3.7 yields

$$\mathrm{Ord}_{\theta\downarrow 0} \min_{s\in\mathcal{R}_0(\Xi(\theta);p^{k_0})} |t_0+T(\theta)-s|^2 sub^2\, \sigma(P^{k_0})(\cdot) \ge \mathrm{Ord}_{\theta\downarrow 0}\, h_1(\cdot;p^{k_0})^{1/2},$$

where $(\cdot = (t_0+T(\theta), -a_1^{k_0}(t_0+T(\theta),\Xi(\theta))/3, \Xi(\theta)$. The repetition of the above argument and (2.26) prove the assertion (ii). □

3.3. The double characteristic factors

Fix j and k_0 so that $1\le j\le N_0$, $1\le k_0\le r(j)$ and $m(j,k_0)=2$. In this subsection we omit the subscript j and the superscript j in the same manner as in §3.1. We also omit the superscript k_0 until Lemma 3.14. Write

$$\begin{aligned}&p(t,\tau,\xi)\,(=p^{j,k_0}(t,\tau,\xi)) = \tau^2 + a_1(t,\xi)\tau + a_2(t,\xi),\\ &\hat{p}(t,\tau,\xi) = p(t,\tau-a_1(t,\xi)/2,\xi) = \tau^2 - \hat{a}_2(t,\xi),\\ &P(t,\tau,\xi) = p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi),\end{aligned}$$

where $q_0(t,\tau,\xi)$ is positively homogeneous of degree 1 in (τ,ξ) for $|\xi|\ge 1$ and

$$\begin{aligned}&\hat{a}_2(t,\xi) = a_1(t,\xi)^2/4 - a_2(t,\xi)\ (\ge 0),\\ &q_0(t,\tau,\xi) \equiv b_0(t,\xi)\tau + b_1(t,\xi) \in \mathcal{S}_{1,0}^{1,0}([0,\delta_1]\times((\overline{\Gamma}\cup(-\overline{\Gamma}))\setminus\{0\})),\\ &q_1(t,\tau,\xi) \equiv c_0(t,\xi)\tau + c_1(t,\xi) \in \mathcal{S}_{cl}^{1,-1}([0,\delta_1]\times((\overline{\Gamma}\cup(-\overline{\Gamma}))\setminus\{0\})).\end{aligned}$$

Here we assume that $P(t,\tau,\xi)\,(=P^{k_0}(t,\tau,\xi))$ is defined for $\xi\in(-\overline{\Gamma})\setminus\{0\}$ as stated in §3.1. We also write

$$\begin{aligned}&\hat{q}_0(t,\tau,\xi) = q_0(t,\tau-a_1(t,\xi)/2,\xi) = b_0(t,\xi)\tau + \hat{b}_1(t,\xi),\\ &\hat{b}_1(t,\xi) = b_1(t,\xi) - a_1(t,\xi)b_0(t,\xi)/2.\end{aligned}$$

Note that

$$h_1(t,\tau-a_1(t,\xi)/2,\xi;p) = h_1(t,\tau,\xi;\hat{p}) = 2\tau^2 + 2\hat{a}_2(t,\xi).$$

Let $t_0\in[0,\delta_1/2]$, $\xi^0\in\Gamma\cap S^{n-1}$ and $\theta_0>0$, and let $T(\theta), \Xi_l(\theta)\in C^\infty((0,\theta_0])\cap C([0,\theta_0])$ ($1\le l\le n$) be real-valued functions satisfying the condition (T,Ξ).

(I) The case where $\hat{a}_2(t, \Xi(\theta)) \not\equiv 0$ in (t, θ).

Applying the Weierstrass preparation theorem, we can write

$$\hat{a}_2(t_0 + t, \Xi(\theta)) = \theta^{l_0} d(t, \theta) \prod_{i=1}^{n_0} (t - t_i(\theta)), \quad d(t, \theta) \neq 0$$

for $(t, \theta) \in [-\delta_0, \delta_0] \times [0, \theta_0]$, where $0 < \delta_0 \leq \delta_1 - t_0$ and $t_i(\theta) \equiv t_i(\theta; t_0, \Xi)$. The $t_i(\theta)$ can be expanded into convergent Puiseux series of θ in $[0, \theta_0]$, with a modification of θ_0 if necessary. Note that

$$\mathcal{R}_0(\Xi(\theta)) \supset \{(t_0 + \operatorname{Re} t_i(\theta))_+;\ 1 \leq i \leq n_0\} \,(\equiv \mathcal{R}_0(\Xi(\theta); p)) \quad (\theta \in (0, \theta_0]).$$

(II) The case where $\hat{a}_2(t, \Xi(\theta)) \equiv 0$ in (t, θ).

We have $\hat{p}(t, \tau, \Xi(\theta)) = \tau^2$, and put

$$\mathcal{R}_0(\Xi(\theta); p) = \emptyset \,(\subset \mathcal{R}_0(\Xi(\theta))), \quad n_0 = 0 \quad \text{and} \quad l_0 = \infty.$$

Now we define

$$\begin{aligned}
&\hat{\mu} \,(\equiv \hat{\mu}(t_0, \xi^0, T, \Xi)) = (\operatorname{Ord}_{\theta \downarrow 0} \hat{a}_2(t_0 + T(\theta), \Xi(\theta)))/2, \\
&\mu_1 \,(\equiv \mu_1(t_0, \xi^0, T, \Xi)) \\
&\ = \operatorname{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \alpha(t_0 + T(\theta), \Xi(\theta)), \\
&\delta \,(\equiv \delta(t_0, \xi^0, T, \Xi)) = \operatorname{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \\
&(= \max_{1 \leq i \leq n_0} \operatorname{Ord}_{\theta \downarrow 0} |t_0 + T(\theta) - (t_0 + \operatorname{Re} t_i(\theta))_+|),
\end{aligned}$$

where

$$\alpha(t, \xi) = \hat{b}_1(t, \xi) + i \partial_t a_1(t, \xi)/2$$

and $\delta = 0$ in the case (II).

Proposition **3.12.** *If*

$$\mu_1 < \hat{\mu}, \tag{3.74}$$

then the Cauchy problem (CP) *is not* C^∞ *well-posed or* (CP) *does not have finite propagation property.*

Corollary **3.13.** *Assume that the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma \cap S^{n-1})$. *Then we have*

$$\hat{\mu}(t_0, \xi^0, T, \Xi) \leq \mu_1(t_0, \xi^0, T, \Xi)$$

if $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T, Ξ).

In the rest of this subsection we shall prove Proposition 3.12. Assume that (3.74) is satisfied. Then we have $\delta \le \mu_1 < \infty$ and $0 < \hat{\mu}\,(\le \infty)$. There is $c_0 > 0$ such that

$$\min_{1\le i\le n_0} |t_0 + T(\theta) - (t_0 + \operatorname{Re} t_i(\theta))_+| \ge c_0\theta^{\delta} \quad \text{for } \theta \in [0, \theta_0].$$

In the case (II) we may take $c_0 = 1$ since $n_0 = 0$. For $v \in \mathbf{R}$ we define $T_v(\theta)$ by (3.49). In the case (I) we have

$$\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{2\hat{\mu}}(d(v) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $d(v) \not\equiv 0$ is a polynomial of v with real coefficients. It is obvious that

$$\begin{aligned} &d(v) > 0 \quad \text{for } v \in [-c_0/2, c_0/2],\\ &\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta))^{1/2} = \theta^{\hat{\mu}}(\sqrt{d(v)} + o(1))\\ &\qquad\qquad\qquad \text{uniformly in } v \in [-c_0/2, c_0/2] \text{ as } \theta \downarrow 0. \end{aligned}$$

Noting that $\alpha(t, \Xi(\theta)) \not\equiv 0$ in (t, θ), we write

$$\alpha(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_1 - \delta}(d_1(v) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $\tilde{\mu}_1 \in \mathbf{Q}$ and $d_1(v)\,(\not\equiv 0)$ is a polynomial of v. There are $v_0 \in (c_0/4, c_0/2)$ and $s_0 > 0$ such that $I_0 \equiv [v_0 - s_0, v_0 + s_0] \subset [c_0/4, c_0/2]$ and

$$d_1(v) \ne 0 \quad \text{for } v \in I_0.$$

We replace $T(\theta)$ by $T_{v_0}(\theta)$. We note that $\delta = \tilde{\mu}_1 = 0$ and $c_0 = 1$ if $\hat{\mu} = \infty$. Then we can assume that $I_0 = [-s_0, s_0]$, $\mu_1 = \tilde{\mu}_1$ and $\mu_1 < \hat{\mu}$. Similarly, we make an asymptotic change of variables as (3.50), where $\delta' \in (0,1)$, $\kappa > 0$ and $\delta'\kappa < 1$. Let K be a compact neighborhood of $(t_0, 0)$ in $\mathbf{R} \times \mathbf{R}^n$, and choose $\rho_0 > 0$ so that (3.52) is satisfied. Define

$$P_\rho(s, \sigma, \eta) = P(t(s;\rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta),$$

and put

$$\begin{aligned} &\Phi(s, y; \rho) = -\rho^{1-\delta\kappa}\int_0^s a_1(t(u;\rho), \Xi(\rho^{-\kappa}))\, du/2 + \rho^{\delta'\kappa} y \cdot \Xi(\rho^{-\kappa}),\\ &E(s, y; \rho, \varepsilon, \nu_0, \varphi) = \exp[i\varepsilon\Phi(s, y; \rho) + i\rho^{\nu_0}\varphi(s;\rho)], \end{aligned}$$

where $\varphi(s;\rho)\,(\in C^\infty([-s_0, s_0])$ for $\rho \ge \rho_0)$ satisfies

$$|\partial_s^l \varphi(s;\rho)| \le C_l \quad \text{for } l \in \mathbf{Z}_+ \text{ and } (s;\rho^{-1}) \in [-s_0, s_0] \times (0, \rho_0^{-1}],$$

$\varepsilon = \pm 1$ and $\nu_0 > 0$. A simple calculation yields

$$\begin{aligned}
&\widetilde{P}(s, D_s; \rho, E)u(s) \equiv E(s, y; \rho, \nu_0, \varphi)^{-1} P_\rho(s, D_s, D_y)(E(s, y; \rho, \nu_0, \varphi)u(s)) \\
&= E(s, 0; \rho, \nu_0, \varphi)^{-1} P(t(s; \rho), \rho^{\delta\kappa} D_s, \varepsilon\rho\Xi(\rho^{-\kappa}))(E(s, 0; \rho, \nu_0, \varphi)u(s)) \\
&= [\rho^{2\delta\kappa+2\nu_0}(\partial_s\varphi(s;\rho))^2 + \varepsilon\rho^{1-\mu_1\kappa+\delta\kappa}(\rho^{\mu_1-\delta\kappa}\alpha(\cdot)) + 2\rho^{2\delta\kappa+\nu_0}(\partial_s\varphi)D_s + \rho^{2\delta\kappa}D_s^2 \\
&\quad - \rho^{2-2\hat{\mu}\kappa}(\rho^{2\hat{\mu}\kappa}\hat{a}_2(\cdot)) - i\rho^{2\delta\kappa+\nu_0}(\partial_s^2\varphi) + \rho^{\delta\kappa+\nu_0}b_0(\cdot)(\partial_s\varphi + \rho^{-\nu_0}D_s) \\
&\quad + c_0(t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa})))(-\varepsilon\rho a_1(\cdot)/2 + \rho^{\delta\kappa+\nu_0}\partial_s\varphi + \rho^{\delta\kappa}D_s) \\
&\quad + c_1(t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa}))]u(s),
\end{aligned}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. We choose

$$\nu_0 = (1 - \mu_1\kappa - \delta\kappa)/2, \quad \kappa = (\delta + \min\{\mu_1 + 1, \hat{\mu}\})^{-1}.$$

Then we have

$$\begin{aligned}
&2\delta\kappa + 2\nu_0 = 1 - \mu_1\kappa + \delta\kappa, \quad 1 - \delta\kappa = \min\{\mu_1 + 1, \hat{\mu}\}\kappa, \\
&\nu_0 = \min\{1, \hat{\mu} - \mu_1\}\kappa/2 \ (> 0), \quad \nu_0 \le 1/2, \\
&2\delta\kappa + 2\nu_0 - (2 - 2\hat{\mu}\kappa) = 2\nu_0 - 2\min\{1 + \mu_1 - \hat{\mu}, 0\}\kappa \ge 2\nu_0.
\end{aligned}$$

Put

$$\varphi(s;\rho) = \int_0^s [-\varepsilon(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/2}\, du.$$

Here we have chosen $\varphi(s;\rho)$ and $\varepsilon = \pm 1$ so that

$$\mathrm{Im}[-\varepsilon(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/2} < 0.$$

Writing $u(s;\rho^{-1})$ as (3.56), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{2(\partial_s\varphi(s;\rho))D_s - i(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)\}u_k(s;\rho^{-1}) \\
&+ \{D_s^2 - \rho^{2\min\{1+\mu_1-\hat{\mu},0\}}(\rho^{2\hat{\mu}\kappa}\hat{a}_2(\cdot)) + \rho^{-\delta\kappa}b_0(\cdot)D_s \\
&\quad + \rho^{-2\delta\kappa}c_0(\cdots)(-\varepsilon\rho a_1(\cdot)/2 + \rho^{\delta\kappa+\nu_0}\partial_s\varphi + \rho^{\delta\kappa}D_s) \\
&\quad + \rho^{-2\delta\kappa}c_1(\cdots)\}u_{k-1}(s;\rho^{-1}) \quad (\ k = 0, 1, 2, \cdots),
\end{aligned}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$ and $(\cdots) = (t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa}))$. Note that, with some $C_\mu > 0$,

$$|(\rho^{\delta\kappa}\partial_s)^\mu c_l(t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa}))| \le C_\mu\rho^{l-1} \quad (\ l = 0, 1).$$

Therefore, applying the same argument as in §3.2 we can prove Proposition 3.12. The same argument as in the proof of Lemma 3.11 and Proposition 3.12 prove the following

Lemma **3.14.** *Assume that* $1 \le k_0 \le r$ *and* $m(k_0) = 2$, *and that* (CP) *is* C^∞ *well-posed and finite propagation property. Let* $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma \cap S^{n-1})$, *and let* $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T, Ξ). *Then we have*

$$\begin{aligned}
&\mathrm{Ord}_{\theta\downarrow 0} \min_{s\in\mathcal{R}_0(\Xi(\theta);p^{k_0})} |t_0 + T(\theta) - s| \\
&\qquad \times \mathit{sub}\ \sigma(P)(t_0 + T(\theta), -a_1^{k_0}(t_0 + T(\theta), \Xi(\theta))/2, \Xi(\theta)) \\
&\ge \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-1}(t_0 + T(\theta), -a_1^{k_0}(t_0 + T(\theta), \Xi(\theta))/2, \Xi(\theta))^{1/2}.
\end{aligned}$$

3.4. Proof of Theorem 1.3

Let $n = 2$, and let $(t_0, \xi^0) \in (0, \infty) \times S^1$. Let $a(t, \xi)$ and $b(t, \xi)$ be real analytic functions defined in a conic neighborhood $\mathcal{C}$ of (t_0, ξ^0). We assume that $a(t_0, \xi^0) = 0$, $a(t, \xi) \ge 0$, $a(t, \xi) \not\equiv 0$ in $\mathcal{C}$ and $a(t, \xi)$ and $b(t, \xi)$ are positively homogeneous in ξ. Choose $\boldsymbol{e} \in S^1$, $\delta \equiv \delta(t_0, \xi^0) > 0$ and $\theta_0 \equiv \theta(t_0, \xi^0) > 0$ so that $\boldsymbol{e} \perp \xi^0$, and

$$\{(t, \widetilde{\Xi}_0(\theta));\ (t_0 - \delta)_+ \le t \le t_0 + \delta \text{ and } |\theta| \le \theta_0\} \subset \mathcal{C},$$

where $\widetilde{\Xi}_0(\theta) \equiv \widetilde{\Xi}_0(\theta; \xi_0, \boldsymbol{e}) = (\xi^0 + \theta\boldsymbol{e})/|\xi^0 + \theta\boldsymbol{e}|$. We write

$$a^0(t, \theta) = a(t, \widetilde{\Xi}_0(\theta)), \quad b^0(t, \theta) = b(t, \widetilde{\Xi}_0(\theta)).$$

Then we have

$$a^0(t, \theta) = \sum_{k=l_0}^{\infty} a_k^0(t)\theta^k, \quad a_{l_0}^0(t) \not\equiv 0,$$

where $l_0 \in \mathbf{Z}_+$. By the Weierstrass preparation theorem there are $r_0 \in \mathbf{Z}_+$, a real analytic function $c^0(t, \theta)$ defined in $[0, \theta_0]$, real-valued continous functions $\tau_k^0(\theta)$ and $\sigma_k^0(\theta)$ ($1 \le k \le r_0$) defined in $[0, \theta_0]$ such that $c^0(t, \theta) > 0$, $\tau_k^0(0) = \sigma_k^0(0) = 0$ ($1 \le k \le r_0$), the $\tau_k^0(\theta)$ and $\sigma_k^0(\theta)$ can be expanded into convergent Puiseux series in $[0, \theta_0]$,

$$\begin{aligned}
&\tau_1^0(\theta) \le \tau_2^0(\theta) \le \cdots \le \tau_{r_0}^0(\theta), \quad \sigma_k^0(\theta) \ge 0 \ (1 \le k \le r_0) \\
&a^0(t, \theta) = \theta^{l_0} c^0(t, \theta) \prod_{k=1}^{r_0} \{(t - t_0 - \tau_k^0(\theta))^2 + \sigma_k^0(\theta)\} \quad (\theta \in [0, \theta_0]),
\end{aligned}$$

with a modification of θ_0 if necessary, where $a^0(t, \theta) = \theta^{l_0} c^0(t, \theta)$ if $r_0 = 0$. Define

$$d^0(t, \theta) = \begin{cases} \theta^{l_0} \displaystyle\sum_{k=1}^{r_0} \prod_{l \ne k} \{(t - t_0 - \tau_l^0(\theta))^2 + \sigma_l^0(\theta)\} & \text{if } r_0 > 1, \\ \theta^{l_0} & \text{if } r_0 \le 1, \end{cases} \tag{3.75}$$

$$\mathcal{R}(\theta; a^0) = \begin{cases} \{t_0 + \tau_k^0(\theta) + i\sqrt{\sigma_k^0(\theta)};\ 1 \le k \le r_0\} & \text{if } r_0 \ge 1, \\ \emptyset & \text{if } r_0 = 0. \end{cases}$$

LEMMA **3.15.** *There is $C > 0$ such that*

$$C^{-1} \min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda| \sqrt{d^0(t, \theta)} \le \sqrt{a^0(t, \theta)}$$
$$\le C \min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda| \sqrt{d^0(t, \theta)}$$

for $t \in [(t_0 - \delta)_+, t_0 + \delta]$ and $\theta \in [0, \theta_0]$, with modifications of δ and θ_0 if necessary, where $\min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda| = 1$ if $r_0 = 0$.

PROOF. When $r_0 = 0$, the lemma is trivial. Assume that $r_0 \ge 1$, and fix $(t, \theta) \in [(t_0 - \delta)_+, t_0 + \delta] \times [0, \theta_0]$. We choose $\nu_0 \in \mathbf{N}$ so that $1 \le \nu_0 \le r_0$ and

$$\min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda|^2 = (t - t_0 - \tau_{\nu_0}^0(\theta))^2 + \sigma_{\nu_0}^0(\theta).$$

Then we have

$$\min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda|^2 d^0(t, \theta) = ((t - t_0 - \tau_{\nu_0}^0(\theta))^2 + \sigma_{\nu_0}^0(\theta)) d^0(t, \theta)$$
$$\le r_0 a^0(t, \theta) \le r_0 \min_{\lambda \in \mathcal{R}(\theta; a^0)} |t - \lambda|^2 d^0(t, \theta).$$

□

Let $1 \le k \le r_0$. Suppose that $b^0(t, \theta) \not\equiv 0$ in (t, θ). We note that $t_0 + \tau_k^0(\theta) \ge 0$ if $0 < \theta \ll 1$, since $t_0 > 0$. Applying the same argument as in §2 of [10], we can write

$$b^0(t_0 + \tau_k^0(\theta) + t, \theta) \sim \sum_{l=0}^{\infty} \beta_{k,l}(t) \theta^{\nu_k + l/L}, \quad \beta_{k,0}(t) \not\equiv 0,$$

where $L \in \mathbf{N}$ and $\nu_k \in \mathbf{Q}$. We define the Newton polygon $\Gamma_{b^0,k}^h$ of $t^h b^0(t_0 + \tau_k^0(\theta) + t, \theta)$ for $h = 0, 1, 2$ by

$$\Gamma_{b^0,k}^h = ch\Big[\bigcup_{l \ge 0,\, \mu_{k,l} < \infty} (\{(\nu_k + l/L, h + \mu_{k.l})\} + [0, \infty)^2)\Big],$$

where

$$\mu_{k,l} = \mathrm{Ord}_{t \downarrow 0}\, \beta_{k,l}(t)$$

and $ch[A]$ denotes the convex hull of A. If $b^0(t, \theta) \equiv 0$ in (t, θ), we define $\Gamma_{b^0,k}^h = \emptyset$ (see, also, §§2 and 5 of [10]). We also denote by $\Gamma_{a^0,k}$ the Newton polygon of $a^0(t_0 + \tau_k^0(\theta) + t, \theta)$.

Lemma **3.16** (Lemma 2.2 of [10])**.** *Fix* $h \in \{0,1,2\}$*. The following two conditions* (i) *and* (ii) *are equivalent:*

(i) *If* $T(\theta)$ *is a real valued continuous function defined in* $[0,\theta_0]$*,* $T(\theta) \in C^\infty((0,\theta_0])$*,* $T(0)=0$*,* $t_0+T(\theta)>0$ *for* $\theta \in (0,\theta_0]$ *and* $T(\theta)$ *can be expanded into a formal Puiseux series, then*

$$\begin{aligned}&\mathrm{Ord}_{\theta\downarrow 0}\{\min_{1\le k\le r_0} |T(\theta)-\tau_k^0(\theta)|^h|b^0(t_0+T(\theta),\theta)|\}\\ &\ge \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a^0(t_0+T(\theta),\theta)}.\end{aligned}$$

(ii) $2\Gamma^h_{b^0,k} \subset \Gamma_{a^0,k} \quad (\ 1\le k\le r_0) \quad$ (see, also, Lemma 3.3 of [12]).

Lemma **3.17.** *Fix* $h \in \{0,1,2,\}$*. Assume that*

$$2\Gamma^h_{b^0,k} \subset \Gamma_{a^0,k} \quad (\ 1\le k\le r_0).$$

Then there is $C>0$ *such that*

$$\min_{1\le k\le r_0} |t-(t_0+\tau_k^0(\theta))|^h|b^0(t,\theta)| \le C\sqrt{a^0(t,\theta)}$$
$$\textit{for } t \in [(t_0-\delta)_+, t_0+\delta] \textit{ and } \theta \in [0,\theta_0],$$

with modifications of δ *and* θ_0 *if necessary.*

We proved Lemma 3.17 with $h=1$ in §5 of [10]. Lemma 3.17 with $h=0,2$ can be proved by the same arguments as in §5 of [10].

We assume that the Cauchy problem (CP) is C^∞ well-posed and has finite propagation property. We factorize $p(t,\tau,\xi)$ as (2.3):

$$p(t,\tau,\xi)=\prod_{k=1}^{r(j)} p^{j,k}(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in [0,\delta_1]\times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1})$$

($1 \le j \le N_0$). Fix j so that $1\le j\le N_0$. Assume that $1\le k_0 \le r(j)$ and $m(j,k_0)=3$. Until the end of this subsection we omit the subscript j and the superscript j in the same manner as in §3.1. Now assume that $\hat{a}_2^{k_0}(t,\xi) \not\equiv 0$ in (t,ξ). It follows from (3.11), (3.47), Corollary 3.7 and Lemmas 3.16 and 3.17 that (3.6) and (2.19) with $\mathcal{R}(\xi)$ replaced by $\mathcal{R}_0(\xi)$ hold for $k=k_0$.

Lemma **3.18.** *Let* $(t_0,\xi^0) \in (0,\delta_1/2)\times(\Gamma\cap S^{n-1})$*, and let* $T(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ *be a real-valued function satisfying the following:*

$$T(0)=0, \quad t_0+T(\theta)>0 \textit{ for } \theta\in[0,\theta_0] \textit{ and}$$
$$T(\theta) \textit{ can be expanded into a formal Puiseux series.}$$

Then we have

$$(3.76)\quad \mathrm{Ord}_{\theta\downarrow 0}\{\min_{s\in\mathcal{R}_0(\widetilde{\Xi}_0;p^{k_0})}|t_0+T(\theta)-s| \\ \times\, sub\ \sigma(P^{k_0})(t_0+T(\theta),(\hat{a}_3^{k_0}(\cdot)/2)^{1/3}-a_1^{k_0}(\cdot)/3,\widetilde{\Xi}_0(\theta))\hat{a}_2^{k_0}(\cdot)\} \\ \geq \mathrm{Ord}_{\theta\downarrow 0}(D_3^{k_0}(\cdot)\hat{a}_2^{k_0}(\cdot))^{1/2},$$

where $(\cdot)=(t_0+T(\theta),\widetilde{\Xi}_0(\theta))$.

Proof. (3.42) yields

$$(\hat{a}_2^{k_0}(t,\xi)/3)h_2(t,(\hat{a}_3^{k_0}(t,\xi)/2)^{1/3},\xi;\hat{p}^{k_0})^{1/2}\leq (D_3^{k_0}(t,\xi)(\hat{a}_2^{k_0}(t,\xi)/3))^{1/2}.$$

Therefore, the lemma easily follows from Corollary 3.7, (3.11) and (3.30) – (3.32). □

We may assume that $D_3^{k_0}(t,\xi)\not\equiv 0$ in (t,ξ). Indeed, if $D_3^{k_0}(t,\xi)\equiv 0$ in (t,ξ), then Lemma 3.18 implies that $sub\ \sigma(P^{k_0})(t,(\hat{a}_3^{k_0}(t,\xi)/2)^{1/3}-a_1^{k_0}(t,\xi)/3,\xi)\equiv 0$ in (t,ξ) and, therefore, (3.5) holds. Taking $a^0(t,\theta)=D_3^{k_0}(t,\widetilde{\Xi}_0(\theta))\hat{a}_2^{k_0}(t,\widetilde{\Xi}_0(\theta))$ in Lemma 3.15, Lemma 3.15 implies that (3.5) are valid if and only if

$$|(\hat{a}_2^{k_0}(\cdot)sub\ \sigma(P^{k_0}(t,(\hat{a}_3^{k_0}(\cdot)/2)^{1/3}-a_1^{k_0}(\cdot)/3,\widetilde{\Xi}_0(\theta))^2|\leq d^0(t,\theta)$$

for each $(t_0,\xi^0)\in[0,\delta_1]\times S^{n-1}$ with $D_3^{k_0}(t_0,\xi^0)=0$ and $(t,\theta)\in[(t_0-\delta)_+,t_0+\delta]\times[-\theta_0,\theta_0]$, where $(\cdot)=(t,\widetilde{\Xi}_0(\theta))$ and $d^0(t,\theta)$ is defined by (3.75), since $\mathcal{R}(\theta;a^0)=\mathcal{R}_0(\widetilde{\Xi}_0(\theta);p^{k_0})$. Choose $L\in\mathbf{N}$ so that $\tau_l^0(s^L)+i\sqrt{\sigma_l^0(s^L)}$ $(1\leq l\leq r_0)$ are real analytic in a neighborhood of $s=0$. We put

$$\tilde{d}(t,s)=d^0(t,s^L),\quad \tilde{a}(t,s)=\hat{a}_3^{k_0}(t,\widetilde{\Xi}_0(s^L))/2$$

which are real analytic in (t,s). Moreover, $\tilde{d}(t,s)$ is a polynomial of t. Note that $\tilde{d}(t,s)$ depends on (t_0,ξ^0). It follows from Hironaka's resolution theorem that for each $(t_0,\xi^0)\in(0,\delta_1/2]\times S^{n-1}$ with $D_3^{k_0}(t_0,\xi^0)=0$ there are an open neighborhood $U(t_0)$ of $(t,s)=(t_0,0)$ in $(0,\delta_1]\times\mathbf{R}$, a real analytic manifold $\widetilde{U}(t_0)$, a proper analytic mapping $\varphi\equiv\varphi(t_0):\widetilde{U}(t_0)\ni\tilde{u}\mapsto\varphi(\tilde{u})\,(\equiv\varphi(\tilde{u};t_0))\in U(t_0)$ satisfying the following:

(i) $\varphi:\widetilde{U}(t_0)\setminus\widetilde{A}\to U(t_0)\setminus A$ is an isomorphism, where $A=\{(t,s)\in U(t_0);\ \tilde{a}(t,s)=0\}$ and $\widetilde{A}=\varphi^{-1}(A)$.

(ii) For each $p\in\widetilde{U}(t_0)$ there are local analytic coordinates $X\,(\equiv X^p)=(X_1,X_2)\,(=(X_1^p,X_2^p))$ centered at p, $\kappa_1,\kappa_2\in\mathbf{Z}_+$, a neighborhood

$\widetilde{U}(t_0;p)$ of p and a real analytic function $e(X)$ in $\widetilde{V}(t_0;p)$ such that $e(X) \neq 0$ in $\widetilde{V}(t_0;p)$ and

$$\tilde{a}(\varphi(\tilde{u})) = e(X(\tilde{u}))X_1(\tilde{u})^{\kappa_1}X_2(\tilde{u})^{\kappa_2} \quad (\ \tilde{u} \in \widetilde{U}(t_0;p)),$$

where $\widetilde{V}(t_0;p) = \{X(\tilde{u});\ \tilde{u} \in \widetilde{U}(t_0;p)\}$ (see [1]).

Define $\tilde{\varphi}\,(\equiv \tilde{\varphi}(t_0;p)) : \widetilde{V}(t_0;p) \to U(t_0)$ by $\tilde{\varphi}(X(\tilde{u}))\,(\equiv \tilde{\varphi}(X^p(\tilde{u});t_0,p)) = \varphi(\tilde{u})\,(\equiv \varphi(\tilde{u};t_0))$ for $\tilde{u} \in \widetilde{U}(t_0;p)$. Then we have

$$\tilde{a}(\tilde{\varphi}(X)) = e(X)X_1^{\kappa_1}X_2^{\kappa_2} \quad (\ X \in \widetilde{V}(t_0;p)).$$

Putting $X_l = \widetilde{X}_l^3$ ($l = 1, 2$), we have

$$\tilde{a}(\varphi^0(\widetilde{X}))^{1/3} = e(\widetilde{X}_1^3, \widetilde{X}_2^3)^{1/3}\widetilde{X}_1^{\kappa_1}\widetilde{X}_2^{\kappa_2} \quad (\ \widetilde{X} \in V^0(t_0;p)),$$

where $\varphi^0(\widetilde{X}) = \tilde{\varphi}(\widetilde{X}_1^3, \widetilde{X}_2^3)$ and $V^0(t_0;p) = \{\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2);\ (\widetilde{X}_1^3, \widetilde{X}_2^3) \in \widetilde{V}(t_0;p)\}$. Put $U(t_0;p) = \{\varphi(\tilde{u});\ \tilde{u} \in \widetilde{U}(t_0;p)\}$, $\tilde{a}^0(\widetilde{X}) = \tilde{a}(\varphi^0(\widetilde{X}))^{1/3}$ and $\varphi^0(\widetilde{X}) = (t(\widetilde{X}), s(\widetilde{X}))$. Then

$$\begin{aligned}(3.77) \qquad & \min\{\min_{v \in \mathcal{R}_0(\widetilde{\Xi}_0(s^L);p^{k_0})} |t - v|, 1\} \\ & \qquad \times |sub\ \sigma(P^{k_0})(t, (\hat{a}_3^{k_0}(\cdot)/2)^{1/3} - a_1^{k_0}(\cdot)/3, \widetilde{\Xi}_0(s^L))| \\ & \le Ch_2(t, (\hat{a}_3^{k_0}(\cdot)/2)^{1/3}, \widetilde{\Xi}_0(s^L); \hat{p}^{k_0})^{1/2} \quad \text{for } (t,s) \in U(t_0;p)\end{aligned}$$

if and only if

$$(3.78) \qquad |B(\widetilde{X})^2| \le C\tilde{d}(t(\widetilde{X}), s(\widetilde{X})) \quad \text{for } \widetilde{X} \in V^0(t_0;p),$$

where $(\cdot) = (t, \widetilde{\Xi}_0(s^L))$ and

$$\begin{aligned}& B(\widetilde{X}) = \hat{a}_2^{k_0}(\cdots)sub\ \sigma(P^{k_0})(t(\widetilde{X}), \tilde{a}^0(\widetilde{X}) - a_1^{k_0}(\cdots), \widetilde{\Xi}_0(s(\widetilde{X})^L)), \\ & (\cdots) = t(\widetilde{X}), \widetilde{\Xi}_0(s(\widetilde{X})^L)).\end{aligned}$$

Note that $\tilde{d}(t(\widetilde{X}), s(\widetilde{X}))$ and $B(\widetilde{X})$ are real analytic in $V^0(t_0;p)$. Let $\widetilde{X}(\theta)$ be real analytic near $\theta = 0$. Then it follows from (3.76) that

$$\mathrm{Ord}_{\theta\downarrow 0}\,\tilde{d}(t(\widetilde{X}(\theta)), s(\widetilde{X}(\theta)))/2 \le \mathrm{Ord}_{\theta\downarrow 0}\,B(\widetilde{X}(\theta)).$$

Lemmas 3.16 and 3.17 with $b^0(t_0 + t, \theta) = B(\widetilde{X})$, $a^0(t_0 + t, \theta) = \tilde{d}(t(\widetilde{X}), s(\widetilde{X}))$, $(t, \theta) = \widetilde{X}$ and $h = 0$ yield (3.78) and, then, (3.77). Let I be a compact sub-interval of $(0, \delta_1/2]$. Applying compactness argument, we can prove that (3.5) holds with $[0, \delta_1]$ and $\mathcal{R}(\xi)$ replaced by I and $\mathcal{R}_0(\xi; p^{k_0})$, respectively. Therefore, Lemma 3.2 shows that (3.4) holds with $[0, \delta_1]$ and $\mathcal{R}(\xi)$ replaced by I and $\mathcal{R}_0(\xi)$, respectively. Next assume that $\hat{a}_2^{k_0}(t, \xi) \equiv 0$ in (t, ξ). Then (3.11) and (3.12) yield $D_3^{k_0}(t, \xi)\,(\equiv \widehat{D}^{k_0}(t, \xi)) \equiv 0$ in (t, ξ). Similarly, (3.4) and (2.19) hold with $\mathcal{R}(\xi)$ replaced by $\mathcal{R}_0(\xi)$. Let $1 \le j \le N_0$ and $1 \le k_0 \le r(j)$ satisfy $m(j, k_0) = 2$. Applying Corollary 3.13 and the same argument as before, we can prove that (3.33) with $\mathcal{R}(\xi)$ replaced by $\mathcal{R}_0(\xi)$ holds. Since (2.19) holds with $\mathcal{R}(\xi)$ and $[0, \delta_1]$ replaced by $\mathcal{R}_0(\xi)$ and I, respectively, as proved above, Lemma 2.5 proves Theorem 1.3 with $I \subset (0, \delta_1/2]$. The interval $(0, \delta_1/2]$ is determined by the factorization (2.3). So, finally one can prove Theorem 1.3 (with any compact interval $I \subset (0, \infty)$).

3.5. Proof of Theorem 1.4

Assume that the hypotheses of Theorem 1.4 are fulfilled. Let $1 \le j \le N_0$, and let $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma_j \cap S^{n-1})$. We fix $l = 1$ or 2. Let $h(t, \xi)$ be defined in a semi-algebraic set U in $\mathbf{R}^{n+1}$. Then we say that $h(t, \xi)$ is a semi-algebraic function if the graph of $h(t, \xi)$ is a semi-algebraic set. Let $a(t, \xi)$ and $b(t, \xi)$ be semi-algebraic functions defined in a conic neighborhood of (t_0, ξ^0). We assume that $a(t, \xi)$ and $b(t, \xi)$ are positively homogeneous in ξ, $a(t, \xi) \ge 0$ and $a(t_0, \xi^0) = 0$. Choose $\hat{\delta} > 0$ so that

$$D_{\hat{\delta}} \equiv \{(t, \xi);\ |t - t_0|^2 + |\xi - \xi^0|^2 \le \hat{\delta}^2,\ |\xi| = 1 \text{ and } t \ge 0\} \subset [0, \delta_1] \times \Gamma_j.$$

We may assume that $a(t, \xi)$ and $b(t, \xi)$ are defined in $D_{\hat{\delta}}$. Then we say that the condition $(A\text{-}B)_l$ is satisfied if

$(A\text{-}B)_l$ there are $\delta \in (0, \hat{\delta}]$ and $C > 0$ satisfying

$$\min\{\min_{s \in \mathcal{R}_0(\xi)} |t - s|^l, 1\}|b(t, \xi)| \le C\sqrt{a(t, \xi)} \quad \text{for } (t, \xi) \in D_\delta.$$

Lemma **3.19.** *Assume that the condition $(A\text{-}B)_l$ is not satisfied. Then there are $\theta_0 > 0$, $T_l(\theta), \Xi_k^l(\theta) \in C^\infty((0, \theta_0]) \cap C([0, \theta_0])$ ($1 \le k \le n$) such that $T_l(\theta)$ and $\Xi^l(\theta)\,(\equiv (\Xi_1^l(\theta), \cdots, \Xi_n^l(\theta)))$ satisfy the condition (T, Ξ) and*

$$\operatorname{Ord}_{\theta \downarrow 0} \min\{\min_{s \in \mathcal{R}_0(\Xi^l(\theta))} |t_0 + T_l(\theta) - s|^l, 1\}|b(\cdot)| < (\operatorname{Ord}_{\theta \downarrow 0} a(\cdot))/2, \tag{3.79}$$

where $(\cdot) = (t_0 + T_l(\theta), \Xi^l(\theta))$.

Proof. Let $\delta \in (0, \hat{\delta}]$. Define

$$A = \{(t, \xi, y) \in D_\delta \times \mathbf{R};\ y = a(t, \xi)\},$$

$$
\begin{aligned}
B &= \{(t,\xi,y) \in D_\delta \times \mathbf{R};\ y = |b(t,\xi)|^2\},\\
C_l &= \{(t,\xi,y) \in D_\delta \times \mathbf{R};\ y = \min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^{2l}, 1\}\}.
\end{aligned}
$$

It is obvious that A and B are semi-algebraic sets. Put

$$
\begin{aligned}
\Xi_0 &= \{\xi \in S^{n-1};\ |\xi-\xi^0| \le \delta \text{ and } D_M(s_0,\xi) \ne 0 \text{ for some } s_0 \in [0,\infty)\},\\
\Xi_k &= \{\xi \in S^{n-1};\ |\xi-\xi^0| \le \delta,\ D_{M-k+1}(s,\xi) = 0 \text{ for any } s \in [0,\infty) \text{ and}\\
&\qquad D_{M-k}(s_0,\xi) \ne 0 \text{ for some } s_0 \in [0,\infty)\} \quad (\ 1 \le k \le M).
\end{aligned}
$$

Since the $D_k(t,\xi)$ are semi-algebraic, the Ξ_k are semi-algebraic set, $\Xi_\mu \cap \Xi_\nu = \emptyset$ if $\mu \ne \nu$, and

$$
\bigcup_{k=0}^{M} \Xi_k = \{\xi \in S^{n-1};\ |\xi-\xi^0| \le \delta\}.
$$

Choose $\delta' \in (0,1]$ so that

$$
\{t+i\tau \in \mathbf{C};\ t \in [-\delta', t_0+2],\ \tau \in \mathbf{R} \text{ and } |\tau| \le \delta'\} \subset \Omega,
$$

where Ω is the complex neighborhood of $[0,\infty)$ as appears in §1. We define

$$
\begin{aligned}
\mathcal{D}_k =& \{(t,\xi) \in \mathbf{R} \times S^{n-1};\ \xi \in \Xi_k,\ D_{M-k}(t_1+i\tau,\xi) = 0,\ t_1 \in [-\delta', t_0+2],\\
&\ \tau \in [-\delta',\delta'],\ t_2 \ge 0,\ t_2^2 = t_1^2 \text{ and } t = (t_1+t_2)/2\} \quad (\ 0 \le k \le M),\\
\mathcal{D} =& \bigcup_{k=0}^{M} \mathcal{D}_k.
\end{aligned}
$$

Note that $\mathcal{D}_M = \emptyset$. Then we have

$$
\begin{aligned}
C_l =& \{(t,\xi,y) \in D_\delta;\ \text{“}(\hat{s},\xi) \in \mathcal{D} \text{ or } \hat{s} = t-1\text{”},\\
&\ |t-s|^2 \ge |t-\hat{s}|^2 \text{ for any } (s,\xi) \in \mathcal{D} \text{ and } y = |t-\hat{s}|^{2l}\}.
\end{aligned}
$$

This shows that C_l is a semi-algebraic set. Put

$$
\begin{aligned}
\Lambda_l =& \{(\rho,t,\xi,\lambda) \in \mathbf{R}^{n+3};\ \text{there are } y,u,v,w \in \mathbf{R} \text{ satisfying}\\
&\ (t,\xi,y) \in A,\ (t,\xi,u) \in B,\ (t,\xi,v) \in C_l,\ \rho y = 1,\\
&\ w((|t-t_0|^2 + |\xi-\xi^0|^2)\rho u v + 1) = 1 \text{ and } \lambda = \rho u v w\}.
\end{aligned}
$$

Then Λ_l is semi-algebraic and

$$
\begin{aligned}
\Lambda_l =& \{(\rho,t,\xi,\lambda) \in \mathbf{R} \times D_\delta \times \mathbf{R};\ \rho a(t,\xi) = 1, \text{ and}\\
&\ \lambda = \rho \min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^{2l}, 1\}|b(t,\xi)|^2
\end{aligned}
$$

$$\times \left((|t-t_0|^2 + |\xi-\xi^0|^2)\rho \min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^{2l}, 1\}|b(t,\xi)|^2+1\right)^{-1}\}.$$

For $\rho > 0$ we define

$$K(\rho) = \{(t,\xi) \in D_\delta;\ \rho a(t,\xi) = 1\}.$$

Then $K(\rho)$ is compact and there is $\rho_0 > 0$ such that $K(\rho) \neq \emptyset$ for $\rho \geq \rho_0$. Indeed, we can take

$$\rho_0^{-1} = \max\{a(t,\xi);\ (t,\xi) \in D_\delta\},$$

since $a(t_0,\xi^0) = 0$. This yields

$$\{\rho \in \mathbf{R};\ (\rho,t,\xi,\lambda) \in \Lambda_l \text{ for some } (t,\xi,\lambda) \in \mathbf{R}^{n+2}\} \supset \{\rho;\ \rho \geq \rho_0\}.$$

Therefore, we can define

$$f_l(\rho) = \sup\{\lambda;\ (\rho,t,\xi,\lambda) \in \Lambda_l \text{ for some } (t,\xi) \in \mathbf{R}^{n+1}\} \quad \text{for } \rho \geq \rho_0. \tag{3.80}$$

Note that

$$\begin{aligned} &f_l(\rho) \\ &\quad = \max\{\frac{\rho \min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^{2l}, 1\}|b(t,\xi)|^2}{((|t-t_0|^2+|\xi-\xi^0|^2)\rho\min\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|^{2l},1\}|b(t,\xi)|^2+1)}; \\ &\qquad\qquad (t,\xi) \in K(\rho)\}, \end{aligned}$$

since $K(\rho)$ is compact. It follows from Theorem A.2.8 of [3] that there are continuous functions $\widetilde{T}_l(\rho)$, $\widetilde{\Xi}^l(\rho)$ and $\lambda_l(\rho)$ such that $\widetilde{T}_l(\rho)$, $\widetilde{\Xi}^l(\rho)$ and $\lambda_l(\rho)$ can be expanded into convergent Puiseux series for $\rho \gg 1$ and

$$(\rho, t_0 + \widetilde{T}_l(\rho), \widetilde{\Xi}^l(\rho), \lambda_l(\rho)) \in \Lambda_l, \quad f_l(\rho) = \lambda_l(\rho)\ (\geq 0) \tag{3.81}$$

(see, also, [7]). Since the condition(A-B)$_l$ does not hold, there is $\{(t_k,\xi^k)\} \subset D_\delta$ satisfying $(t_k,\xi^k) \to (t_0,\xi^0)$ and

$$\min\{\min_{s\in\mathcal{R}_0(\xi^k)} |t_k-s|^l, 1\}|b(t_k,\xi^k)|/a(t_k,\xi^k)^{1/2} \to \infty \quad \text{as } k \to \infty. \tag{3.82}$$

Put $\delta_k = (|t_k-t_0|^2+|\xi^k-\xi^0|^2)^{1/2}$ and $\rho_k = a(t_k,\xi^k)^{-1}$. Then we have $\delta_k \to 0$ and $\rho_k \to \infty$ as $k \to \infty$. (3.81), together with (3.80) and (3.82), gives

$$\begin{aligned} \lambda_l(\rho_k) &\geq \rho_k \min\{\min_{s\in\mathcal{R}_0(\xi^k)} |t_k-s|^{2l},1\}|b(t_k,\xi^k)|^2 \\ &\quad \times (\delta_k^2\rho_k\min\{\min_{s\in\mathcal{R}_0(\xi^k)} |t_k-s|^{2l},1\}|b(t_k,\xi^k)|^2+1)^{-1} \to \infty \quad \text{as } k\to\infty, \end{aligned}$$

since $\delta_k \to 0$ and $\rho_k \min\{\min_{s\in\mathcal{R}_0(\xi^k)} |t_k - s|^{2l}, 1\}|b(t_k, \xi^k)|^2 \to \infty$ as $k \to \infty$. So we have $\lambda_l(\rho) \to \infty$ as $\rho \to \infty$, which implies that

$$\begin{aligned}
&\min\{\min_{s\in\mathcal{R}_0(\widetilde{\Xi}^l(\rho))} |t_0 + \widetilde{T}_l(\rho) - s|^l, 1\} \\
&\quad \times |b(t_0 + \widetilde{T}_l(\rho), \widetilde{\Xi}^l(\rho))| a(t_0 + \widetilde{T}_l(\rho), \widetilde{\Xi}^l(\rho))^{-1/2} \to \infty, \\
&(\widetilde{T}_l(\rho), \widetilde{\Xi}^l(\rho)) \to (0, \xi^0)
\end{aligned}$$

as $\rho \to \infty$. There is $L \in \mathbf{N}$ such that $\widetilde{\Xi}^l(\rho^L)$ is real analytic in ρ ($\geq \rho_0^{1/L}$). We put $T_l(\theta) = \widetilde{T}_l(\theta^{-L})$ and $\Xi^l(\theta) = \widetilde{\Xi}^l(\theta^{-L})$. Here, if $t_0 + T_l(\theta) \equiv 0$, then we replace $T_l(\theta)$ by $T_l(\theta) + \theta^N$, where $N \gg 1$. We note that $\Xi_k^l(\theta)$ ($1 \leq k \leq n$) are real analytic in $\theta \in [0, \theta_0]$, where $\theta_0 = \rho_0^{-1/L}$. Then we have (3.79). □

First we assume that $1 \leq k_0 \leq r(j)$ and $m(j, k_0) = 3$. We take

$$\begin{cases} a(t,\xi) = h_{m-1}(t, A^{j,k_0}(t,\xi) - a_1^{j,k_0}(t,\xi)/3, \xi), \\ b(t,\xi) = sub\ \sigma(P)(t, A^{j,k_0}(t,\xi) - a_1^{j,k_0}(t,\xi)/3, \xi) \end{cases}$$

and $l = 1$, where

$$\begin{aligned}
&A^{j,k_0}(t,\xi) = \nu^{j,k_0}(t,\xi)(\hat{a}_2^{j,k_0}(t,\xi)/3)^{1/2}, \\
&\nu^{j,k_0}(t,\xi) = \begin{cases} 1 & \text{if } \hat{a}_3^{j,k_0}(t,\xi) \geq 0, \\ -1 & \text{if } \hat{a}_3^{j,k_0}(t,\xi) < 0 \end{cases}
\end{aligned}$$

(see (3.9)). It is easy to see that the coefficients of the polynomial $p^{j,k_0}(t, \tau, \xi)$ of τ are semi-algebraic. It follows from Lemma 3.19 that there are $\theta_0 > 0$, $T(\theta), \Xi_k(\theta) \in C^\infty((0, \theta_0]) \cap C([0, \theta_0])$ ($1 \leq k \leq n$) such that $T(\theta)$ and $\Xi(\theta)$ ($\equiv (\Xi_1(\theta), \cdots, \Xi_n(\theta))$) satisfy the condition (T, Ξ) and

$$\begin{aligned}
&\mathrm{Ord}_{\theta\downarrow 0}\{\min\{\min_{s\in\mathcal{R}_0(\Xi(\theta))} |t_0 + T(\theta) - s|, 1\} \\
&\qquad \times sub\ \sigma(P)(t_0 + T(\theta), A^{j,k_0}(\cdot) - a_1^{j,k_0}(\cdot)/3, \Xi(\theta))\} \\
&< \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-1}(t_0 + T(\theta), A^{j,k_0}(\cdot) - a_1^{j,k_0}(\cdot)/3, \Xi(\theta))^{1/2}
\end{aligned}$$

if the condition $(A\text{-}B)_1$ is not satisfied, where $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. Therefore, Lemma 3.11 implies that the condition $(A\text{-}B)_1$ is satisfied if the Cauchy problem (CP) is C^∞ well-posed and has finite propagation property. Next we take

$$\text{(3.83)} \qquad \begin{cases} a(t,\xi) = \hat{a}_2^{j,k_0}(t,\xi), \\ b(t,\xi) = (\partial_\tau sub\ \sigma(P))(t, A^{j,k_0}(t,\xi) - a_1^{j,k_0}(t,\xi)/3, \xi) \end{cases}$$

and $l = 1$. Similarly, we can see that the condition $(A\text{-}B)_1$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (3.83) and (CP) is C^∞ well-posed and has finite propagation property. Let $z^0 = (t_0, \tau_0, \xi^0)$ satisfy $(\partial_\tau^\mu p)(z^0) = 0$ ($\mu = 0, 1, 2$) and $p^{k_0}(z^0) = 0$. We take

$$\begin{cases} a(t,\xi) = h_{m-2}(t, -a_1(t,\xi;z^0)/3, \xi), \\ b(t,\xi) = Q(t, -a_1(t,\xi;z^0)/3, \xi; z^0) \end{cases} \tag{3.84}$$

and $l = 2$. It is easy to see that the coefficients of the polynomials $p(t,\tau,\xi;z^0)$ and $Q(t,\tau,\xi;z^0)$ of τ are semi-algebraic. Similarly, we can see that the condition $(A\text{-}B)_2$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (3.84) and (CP) is C^∞ well-posed and has finite propagation property. This implies that (L-2) for $[0, \delta_1/2]$ is satisfied if (L-1) for $[0, \delta_1/2]$ is satisfied. Now we assume that $1 \le k_0 \le r(j)$ and $m(j, k_0) = 2$. We take

$$\begin{cases} a(t,\xi) = h_{m-1}(t, -a_1^{j,k_0}(t,\xi)/2, \xi), \\ b(t,\xi) = sub\ \sigma(P)(t, -a_1^{j,k_0}(t,\xi)/2, \xi) \end{cases} \tag{3.85}$$

and $l = 1$. Repetition of the above argument and Lemma 3.14 shows that the condition $(A\text{-}B)_1$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (3.85) and (CP) is C^∞ well-posed and has finite propagation property. It follows from the above results and Lemma 2.3 that (3.7), (3.8) and (3.34) hold for $(t,\xi) \in [0, \delta_1/2] \times (\overline{\Gamma}_j \cap S^{n-1})$, since

$$h_2(t, A^{j,k}(t,\xi) - a_1^{j,k}(t,\xi)/3, \xi; p^{j,k}) \approx h_{m-1}(t, A^{j,k}(t,\xi) - a_1^{j,k}(t,\xi)/3, \xi)$$

for $(t,\xi) \in [0, \delta_1] \times (\overline{\Gamma}_j \cap S^{n-1})$ if $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 3$, and

$$2\hat{a}_2^{j,k}(t,\xi)\,(= h_1(t, -a_1^{j,k}(t,\xi)/2, \xi; p^{j,k})) \approx h_{m-1}(t, -a_1^{j,k}(t,\xi)/2, \xi)$$

for $(t,\xi) \in [0, \delta_1] \times (\overline{\Gamma}_j \cap S^{n-1})$ if $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 2$. Therefore, Lemma 2.5 implies that (L-1) for $[0, \delta_1/2]$ is satisfied, which proves Theorem 1.4 with $T = \delta_1/2$. The interval $[0, \delta_1/2]$ is determined by the factorization (2.3). So, finally one can prove Theorem 1.4 (with $I = [0, T]$ for any $T > 0$).

References

[1] Atiyah, M. F., Resolution of singularities and division of distributions, Comm. Pure Appl. Math. **23** (1970), 145–150.

[2] Bierstone, E., Extension of Whitney fields from subanalytic set, Invent. Math. **46** (1978), 277–300.

[3] Hörmander, L., The Analysis of Linear Partial Differential Operators II, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[4] Ivrii, V. Ja. and Petkov, V., Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Uspehi Mat. Nauk. **29** (1971), 3–70. (Russian: English translation in Russian Math. Surveys.)

[5] Kumano-go, H., Pseudo-Differential Operators, MIT Press, Cambridge, 1982.

[6] Seely, R. T., Extension of C^∞ functions defined in a half space, Proc. Amer. Math. Soc. **15** (1964), 625–626.

[7] Svensson, S. L., Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part, Ark. Mat. **8** (1969), 145–162.

[8] Wakabayashi, S., Singularities of solutions of the Cauchy problem for hyperbolic systems in Gevrey classes, Japanese J. Math. **11** (1985), 157–201.

[9] Wakabayashi, S., Remarks on hyperbolic polynomials, Tsukuba J. Math. **10** (1986), 17–28.

[10] Wakabayashi, S., On the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable, J. Math. Soc. Japan **62** (2010), 95–133.

[11] Wakabayashi, S., On the Cauchy problem for a class of hyperbolic operators whose coefficients depend only on the time variable, Tsukuba J. Math. **39** (2015), 121–163.

[12] Wakabayashi, S., On the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time dependent coefficients, Funkcialaj Ekvacioj **63** (2020), 345–418.

[13] Wakabayashi, S., An alternative proof of Ivrii-Petkov's necessary condition for C^∞ well-posedness of the Cauchy problem, located in http://www17.plala.or.jp/wkbysh/index.html.

[14] Wakabayashi, S., Remarks on semi-algebraic functions, located in http://www17.plala.or.jp/wkbysh/index.html.

[15] Wakabayashi, S., Remarks on semi-algebraic functions II, located in http://www17.plala.or.jp/wkbysh/index.html.

Seiichiro Wakabayashi
(professor emeritus)
University of Tsukuba
e-mail: wkbysh@wb3.so-net.ne.jp