Remarks on Propagation of Analytic Singularities and Solvability in the Space of Microfunctions

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1. Introduction

Let P be a linear partial differential operator on $\mathbb{R}^n$ with C^∞ coefficients, and let $x^0 \in \mathbb{R}^n$. In Treves [5] and Yoshikawa [8] it was proved that if P is hypoelliptic at x^0, then there is a neighborhood U of x^0 satisfying the following; for every $f \in C^\infty(U)$ there is $u \in \mathcal{D}'(U)$ such that ${}^tPu = f$ in U. Here ${}^tP$ denotes the transposed operator of P. Hörmander [3] generalized their results (see Theorem 1.2.4 of [3]). Recently Albanese, Corli and Rodino proved in [1] that the result of Treves and Yoshikawa is still valid in the framework of the Gevrey classes and the spaces of ultradistributions. Moreover, Cordaro and Trépreau proved in [2] that Hörmander's result can be generalized in the space of hyperfunctions for partial differential operators with analytic coefficients. In particular, they proved that P is locally solvable at x^0 in the space of hyperfunctions if the coefficients of P are analytic and P is analytic hypoelliptic at x^0. The aim of this article is to microlocalize their results for a pseudodifferential operator $p(x, D)$, *i.e.*, if $\mathcal{U}$ is a bounded open subset of the cosphere bundle $S^*\mathbb{R}^n$ ($\simeq \mathbb{R}^n \times S^{n-1}$) over $\mathbb{R}^n$ and if $p(x, D)$ satisfies

$$\begin{aligned} & f \in L^2(\mathbb{R}^n), \quad WF_A(f) \cap \partial\mathcal{U} = \emptyset, \quad WF_A(p(x,D)f) \cap \mathcal{U} = \emptyset \\ & \quad \Longrightarrow \\ & WF_A(f) \cap \mathcal{U} = \emptyset, \end{aligned}$$

then the transposed operator ${}^tp(x, D)$: $\mathcal{C}(\check{\mathcal{U}}) \to \mathcal{C}(\check{\mathcal{U}})$ is surjective, where $\check{\mathcal{U}} = \{(x, \xi);\ (x, -\xi) \in \mathcal{U}\}$ and $\mathcal{C}(\mathcal{U})$ denotes the space of microfunctions on $\mathcal{U}$.

We shall explain briefly about hyperfunctions, microfunctions and pseudodifferential operators acting on them. For the details we refer to [6]. Let $\varepsilon \in \mathbb{R}$, and denote $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ and $|\xi| = (\sum_{j=1}^n |\xi_j|^2)^{1/2}$. We define

$$\widehat{\mathcal{S}}_\varepsilon := \{v(\xi) \in C^\infty(\mathbb{R}^n);\ e^{\varepsilon\langle\xi\rangle} v(\xi) \in \mathcal{S}\},$$

where $\mathcal{S}$ ($\equiv \mathcal{S}(\mathbb{R}^n)$) denotes the Schwartz space. We introduce the topology to $\widehat{\mathcal{S}}_\varepsilon$ in a natural way. Then the dual space $\widehat{\mathcal{S}}'_\varepsilon$ of $\widehat{\mathcal{S}}_\varepsilon$ can be identified with $\{v(\xi) \in \mathcal{D}'$; $e^{-\varepsilon\langle\xi\rangle} v(\xi) \in \mathcal{S}'\}$, since $\mathcal{D}$ ($= C_0^\infty(\mathbb{R}^n)$) is dense in $\widehat{\mathcal{S}}_\varepsilon$. If $\varepsilon \ge 0$, then $\widehat{\mathcal{S}}_\varepsilon$ is a dense subset of $\mathcal{S}$ and we can define $\mathcal{S}_\varepsilon := \mathcal{F}^{-1}[\widehat{\mathcal{S}}_\varepsilon]$ ($= \mathcal{F}[\widehat{\mathcal{S}}_\varepsilon]$) ($\subset \mathcal{S}$), where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathcal{S}$ (or $\mathcal{S}'$), respectively. For example, $\mathcal{F}[u](\xi) = \int e^{-ix\cdot\xi} u(x)\,dx$ for $u \in \mathcal{S}$, where $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. Let $\varepsilon \ge 0$. We introduce the topology in $\mathcal{S}_\varepsilon$ so that $\mathcal{F} : \widehat{\mathcal{S}}_\varepsilon \to \mathcal{S}_\varepsilon$ is homeomorphic. Denote by $\mathcal{S}'_\varepsilon$ the dual space of $\mathcal{S}_\varepsilon$. Since $\mathcal{S}_\varepsilon$ is dense in $\mathcal{S}$, we can regard $\mathcal{S}'$ as a subspace of $\mathcal{S}'_\varepsilon$. We can define the transposed operators ${}^t\mathcal{F}$ and ${}^t\mathcal{F}^{-1}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$, which map $\mathcal{S}'_\varepsilon$ and $\widehat{\mathcal{S}}'_\varepsilon$ onto $\widehat{\mathcal{S}}'_\varepsilon$ and $\mathcal{S}'_\varepsilon$, respectively. Since $\widehat{\mathcal{S}}_{-\varepsilon} \subset \widehat{\mathcal{S}}'_\varepsilon$ ($\subset \mathcal{D}'$), we can define $\mathcal{S}_{-\varepsilon} = {}^t\mathcal{F}^{-1}[\widehat{\mathcal{S}}_{-\varepsilon}]$, and introduce the topology in $\mathcal{S}_{-\varepsilon}$ so that ${}^t\mathcal{F}^{-1} : \widehat{\mathcal{S}}_{-\varepsilon} \to \mathcal{S}_{-\varepsilon}$ is homeomorphic. $\mathcal{S}'_{-\varepsilon}$ denotes the dual space of $\mathcal{S}_{-\varepsilon}$. We note that $\mathcal{S}'_{-\varepsilon} = \mathcal{F}[\widehat{\mathcal{S}}'_{-\varepsilon}] \subset \mathcal{S}' \subset \mathcal{S}'_\varepsilon$ and $\mathcal{F} = {}^t\mathcal{F}$ on $\mathcal{S}'$. So we also represent ${}^t\mathcal{F}$ by $\mathcal{F}$. Let $\mathcal{A}(\mathbb{C}^n)$ be the space of entire analytic functions on $\mathbb{C}^n$, and let K be a compact subset of $\mathbb{C}^n$. We denote by $\mathcal{A}'(K)$ the space of analytic functionals carried by K, *i.e.*, $u \in \mathcal{A}'(K)$ if and only if (i) $u : \mathcal{A}(\mathbb{C}^n) \ni \varphi \mapsto u(\varphi) \in \mathbb{C}$ is a linear functional, and (ii) for any neighborhood ω of K in $\mathbb{C}^n$ there is $C_\omega \ge 0$ such that $|u(\varphi)| \le C_\omega \sup_{z\in\omega} |\varphi(z)|$ for $\varphi \in \mathcal{A}(\mathbb{C}^n)$. Define $\mathcal{A}'(\mathbb{R}^n) := \bigcup_{K \Subset \mathbb{R}^n} \mathcal{A}'(K)$, $\mathcal{S}_\infty := \bigcap_{\varepsilon\in\mathbb{R}} \mathcal{S}_\varepsilon$, $\mathcal{E}_0 := \bigcap_{\varepsilon>0} \mathcal{S}_{-\varepsilon}$ and $\mathcal{F}_0 := \bigcap_{\varepsilon>0} \mathcal{S}'_\varepsilon$. Here $A \Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\overset{\circ}{B}$ of B. We note that $\mathcal{F}^{-1}[C_0^\infty(\mathbb{R}^n)] \subset \mathcal{S}_\infty$ and that $\mathcal{S}_\infty$ is dense in $\mathcal{S}_\varepsilon$ and $\mathcal{S}'_\varepsilon$ for $\varepsilon \in \mathbb{R}$. For $u \in \mathcal{A}'(\mathbb{R}^n)$ we can define the Fourier transform $\hat{u}(\xi)$ of u by

$$\hat{u}(\xi)\,(= \mathcal{F}[u](\xi)) = u_z(e^{-iz\cdot\xi}),$$

where $z \cdot \xi = \sum_{j=1}^n z_j \xi_j$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. By definition we have $\hat{u}(\xi) \in \bigcap_{\varepsilon>0} \widehat{\mathcal{S}}_{-\varepsilon}$ ($= \mathcal{F}[\mathcal{E}_0]$). Therefore, we can regard $\mathcal{A}'(\mathbb{R}^n)$ as a subspace of $\mathcal{E}_0$, *i.e.*, $\mathcal{A}'(\mathbb{R}^n) \subset \mathcal{E}_0 \subset \mathcal{F}_0$, (see Lemma 1.1.2 of [6]). The space $\mathcal{F}_0$ plays an important role in our treatment as the space $\mathcal{S}'$ does in the framework of C^∞ and distributions. For a bounded open subset X of $\mathbb{R}^n$ we define the space $\mathcal{B}(X)$ of hyperfunctions in X by

$$\mathcal{B}(X) := \mathcal{A}'(\overline{X})/\mathcal{A}'(\partial X),$$

where ∂X denotes the boundary of X.

Let $u \in \mathcal{F}_0$. We define

$$\begin{aligned}&\mathcal{H}(u)(x,x_{n+1}) := (\mathrm{sgn}\ \ x_{n+1})\exp[-|x_{n+1}|\langle D\rangle]u(x)/2\\&(\ = (\mathrm{sgn}\ \ x_{n+1})\mathcal{F}_\xi^{-1}[\exp[-|x_{n+1}|\langle\xi\rangle]\hat{u}(\xi)](x)/2 \in \mathcal{S}'(\mathbb{R}^n))\end{aligned}$$

for $x_{n+1} \in \mathbb{R}\setminus\{0\}$, and

$$\begin{aligned}\mathrm{supp}\ \ u := \bigcap\{F;\ &F \text{ is a closed subset of } \mathbb{R}^n \text{ and there is a real}\\&\text{analytic function } U(x,x_{n+1}) \text{ in } \mathbb{R}^{n+1}\setminus F\times\{0\}\\&\text{such that } U(x,x_{n+1}) = \mathcal{H}(u)(x,x_{n+1}) \text{ for } x_{n+1}\neq 0\}.\end{aligned}$$

We note that supp u coincides with the support of u as a distribution if $u \in \mathcal{S}'$ (see Lemma 1.2.2 of [6]). Let K be a compact subset of $\mathbb{R}^n$. Then $u \in \mathcal{A}'(K)$ if and only if u is an analytic functional and supp $u \subset K$ (see Proposition 1.2.6 of [6]). It follows from Theorem 1.3.3 of [6] that there is $v \in \mathcal{A}'(K)$ satisfying supp $(u-v)\cap K \subset \partial K$, and if $v = v_1$, v_2 are such functionals in $\mathcal{A}'(K)$ we have supp $(v_1 - v_2) \subset \partial K$. Therefore, we can define the restriction map from $\mathcal{F}_0$ to $\mathcal{A}'(K)/\mathcal{A}'(\partial K)$ ($= \mathcal{B}(\overset{\circ}{K})$) which is surjective. For $x^0 \in \mathbb{R}^n$ we say that u is analytic at x^0 if $\mathcal{H}(u)(x,x_{n+1})$ can be continued analytically from $\mathbb{R}^n\times(0,\infty)$ to a neighborhood of $(x^0,0)$ in $\mathbb{R}^{n+1}$. We define

$$\text{sing supp}\ \ u := \{x\in\mathbb{R}^n;\ u \text{ is not analytic at } x\}.$$

Next let $u \in \mathcal{B}(X)$, where X is a bounded open subset of $\mathbb{R}^n$. Then there is $v \in \mathcal{A}'(\overline{X})$ such that the residue class of v is u in $\mathcal{B}(X)$. We define

$$\mathrm{supp}\ \ u := \mathrm{supp}\ \ v \cap X, \quad \text{sing supp}\ \ u := \text{sing supp}\ \ v\cap X.$$

These definitions do not depend on the choice of v. So we say that u is analytic at x^0 if $x^0 \notin$ sing supp u. Let X be an open subset of $\mathbb{R}^n$. We also define $\mathcal{B}(X)$ (see Definition 1.4.5 of [6]). For open subsets U and V of X with $V\subset U$ the restriction map $\rho_V^U : \mathcal{B}(U) \ni u \mapsto u|_V \in \mathcal{B}(V)$ can be defined so that ρ_U^U is the identity mapping and $\rho_W^V\circ\rho_V^U = \rho_W^U$ for open subsets U, V and W of X with $W\subset V\subset U$. By definition we can also define the restriction map from $\mathcal{F}_0$ to $\mathcal{B}(X)$, and we denote by $v|_X$ the restriction of $v\in\mathcal{F}_0$ to $\mathcal{B}(X)$ (or on X). We define the presheaf $\mathcal{B}_X$ by associating $\mathcal{B}(U)$ to every open subset U of X. By definition $\mathcal{B}_X$ is a sheaf on X.

Next we shall define analytic wave front sets and microfunctions.

Definition 1.1. (i) Let $u \in \mathcal{F}_0$. The analytic wave front set $WF_A(u) \subset T^*\mathbb{R}^n \setminus 0$ ($\simeq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$) is defined as follows: $(x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ does not belong to $WF_A(u)$ if there are a conic neighborhood Γ of ξ^0, $R_0 > 0$ and $\{g^R(\xi)\}_{R \geq R_0} \subset C^\infty(\mathbb{R}^n)$ such that $g^R(\xi) = 1$ in $\Gamma \cap \{\langle \xi \rangle \geq R\}$,

$$|\partial_\xi^{\alpha+\tilde{\alpha}} g^R(\xi)| \leq C_{|\tilde{\alpha}|}(C/R)^{|\alpha|} \langle \xi \rangle^{-|\tilde{\alpha}|} \tag{1.1}$$

if $\langle \xi \rangle \geq R|\alpha|$, and $g^R(D)u$ ($= \mathcal{F}^{-1}[g^R(\xi)\hat{u}(\xi)]$) is analytic at x^0 for $R \geq R_0$, where C is a positive constant independent of R.

(ii) Let X be an open subset of $\mathbb{R}^n$, and let $u \in \mathcal{B}(X)$ and $(x^0, \xi^0) \in T^*X \setminus 0$ ($\simeq X \times (\mathbb{R}^n \setminus \{0\})$). Then we say that $(x^0, \xi^0) \notin WF_A(u)$ ($\subset T^*X \setminus 0$) if there are a bounded open neighborhood U of x^0 and $v \in \mathcal{A}'(\overline{U})$ such that $v|_U = u|_U$ in $\mathcal{B}(U)$ and $(x^0, \xi^0) \notin WF_A(v)$

Remark. (i) $WF_A(u)$ for $u \in \mathcal{B}(X)$ is well-defined. Indeed, it follows from Theorem 2.6.5 in [6] that for any $v \in \mathcal{A}'(\mathbb{R}^n)$ with $x^0 \notin \operatorname{supp} v$ there is $R_1 > 0$ such that $g^R(D)v$ is analytic at x^0 if $R \geq R_1$, where $\{g^R(\xi)\}_{R \geq R_0}$ is a family of symbols satisfying (1.1).

(ii) Several remarks on this definition are given in Proposition 3.1.2 of [6].

(iii) From Theorem 3.1.6 in [6] and the results in [4] it follows that our definition of $WF_A(u)$ coincides with the usual definition.

Let $\mathcal{U}$ be an open subset of the cosphere bundle $S^*\mathbb{R}^n$ over $\mathbb{R}^n$, which is identified with $\mathbb{R}^n \times S^{n-1}$. We define

$$\mathcal{C}(\mathcal{U}) := \mathcal{B}(\mathbb{R}^n)/\{u \in \mathcal{B}(\mathbb{R}^n);\ WF_A(u) \cap \mathcal{U} = \emptyset\}.$$

Since $\mathcal{B}$ is a flabby sheaf, we have

$$\mathcal{C}(\mathcal{U}) = \mathcal{B}(U)/\{u \in \mathcal{B}(U);\ WF_A(u) \cap \mathcal{U} = \emptyset\}$$

if U is an open subset of $\mathbb{R}^n$ and $\mathcal{U} \subset U \times S^{n-1}$. Elements of $\mathcal{C}(\mathcal{U})$ are called microfunctions on $\mathcal{U}$. We can define the restriction map $\mathcal{C}(\mathcal{U}) \ni u \mapsto u|_\mathcal{V} \in \mathcal{C}(\mathcal{V})$ for open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n \times S^{n-1}$ with $\mathcal{V} \subset \mathcal{U}$. Let Ω be an open subset of $\mathbb{R}^n \times S^{n-1}$. We define the presheaf $\mathcal{C}_\Omega$ on Ω associating $\mathcal{C}(\mathcal{U})$ to every open subset $\mathcal{U}$ of Ω. Then $\mathcal{C}_\Omega$ is a flabby sheaf (see, *e.g.*, Theorem 3.6.1 of [6]). For each open subset U of $\mathbb{R}^n$ we define the mapping sp: $\mathcal{B}(U) \to \mathcal{C}(U \times S^{n-1})$ such that the residue class in $\mathcal{C}(U \times S^{n-1})$ of $u \in \mathcal{B}(U)$ is equal to $\operatorname{sp}(u)$. We also write $u|_\mathcal{U} = \operatorname{sp}(u)|_\mathcal{U}$ for $u \in \mathcal{B}(U)$ and $v|_\mathcal{U} = \operatorname{sp}(v|_U)|_\mathcal{U}$ for $v \in \mathcal{F}_0$, where $\mathcal{U}$ is an open subset of $U \times S^{n-1}$.

Assume that $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and there are positive constants C_k ($k \geq 0$) such that

$$
\begin{aligned}
(1.2) \qquad & |\partial_\xi^\alpha D_y^{\beta+\tilde{\beta}} \partial_\eta^\gamma a(\xi, y, \eta)| \\
& \leq C_{|\alpha|+|\tilde{\beta}|+|\gamma|} (A/R)^{|\beta|} \langle \xi \rangle^{m_1+|\beta|} \langle \eta \rangle^{m_2} \exp[\delta_1 \langle \xi \rangle + \delta_2 \langle \eta \rangle]
\end{aligned}
$$

if $\alpha, \beta, \tilde{\beta}, \gamma \in (\mathbb{Z}_+)^n$, $\xi, y, \eta \in \mathbb{R}^n$ and $\langle \xi \rangle \geq R|\beta|$, where $D_y = -i\partial_y$, $R \geq 1$, $A \geq 0$, $m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. It should be remarked that some functions satisfying the estimates (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$ are given in Proposition 2.2.3 of [6]. We define pseudodifferential operators $a(D_x, y, D_y)$ and ${}^r a(D_x, y, D_y)$ by

$$
a(D_x, y, D_y)u(x) = (2\pi)^{-n} \mathcal{F}_\xi^{-1} \Big[\int \Big(\int e^{-iy\cdot(\xi-\eta)} a(\xi, y, \eta) \hat{u}(\eta)\, d\eta \Big) dy \Big](x)
$$

and ${}^r a(D_x, y, D_y)u = b(D_x, y, D_y)u$ for $u \in \mathcal{S}_\infty$, respectively, where $b(\xi, y, \eta) = a(\eta, y, \xi)$. Applying the same argument as in the proof of Theorem 2.3.3 of [6] we have the following

Proposition 1.2. *$a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{\varepsilon_2}$ to $\mathcal{S}_{\varepsilon_1}$ and from $\mathcal{S}'_{-\varepsilon_2}$ to $\mathcal{S}'_{-\varepsilon_1}$, respectively, if*

$$
(1.3) \qquad \begin{cases} \nu > 1, \quad \varepsilon_2 - \delta_2 = \nu(\varepsilon_1 + \delta_1)_+, \\ \varepsilon_1 + \delta_1 \leq 1/R, \quad R \geq e\sqrt{n}\nu A/(\nu - 1), \end{cases}
$$

where $c_+ = \max\{c, 0\}$. Similarly, ${}^r a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{-\varepsilon_1}$ to $\mathcal{S}_{-\varepsilon_2}$ and from $\mathcal{S}'_{\varepsilon_1}$ to $\mathcal{S}'_{\varepsilon_2}$, respectively, if (1.3) *is valid.*

Remark. (i) We had a slight improvement of the remark of Theorem 2.3.3 of [6], *i.e.*, we can take $R_1(S, T, \nu) = e\sqrt{n}\nu/(\nu - 1)$ there instead of $R_1(S, T, \nu) = en\nu/(\nu - 1)$ if $n = n' = n''$, $S(y, \xi) = -y \cdot \xi$ and $T(y, \eta) = y \cdot \eta$. This is reflected in the condition (1.3).

(ii) Since for any open sets X_j ($j = 1, 2$) with $X_1 \Subset X_2$ one can construct a symbol $a(\xi, y, \eta)$ satisfying (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$, supp $a \subset \mathbb{R}^n \times X_2 \times \mathbb{R}^n$ and $a(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^n$, one can use the operator $a(D_x, y, D_y)$ instead of cut-off functions.

Definition 1.3. Let Γ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let X be an open subset of $\mathbb{R}^n$. Moreover, let $R_0 \geq 0$.

(i) Let $R_0 \geq 1$, $m, \delta \in \mathbb{R}$ and $A, B \geq 0$, and let $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(x,\xi) \in S^{m,\delta}(R_0, A, B)$ if $a(x,\xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{(\beta+\tilde{\beta})}(x,\xi)| \leq C_{|\tilde{\alpha}|+|\tilde{\beta}|}(A/R_0)^{|\alpha|}(B/R_0)^{|\beta|}\langle\xi\rangle^{m+|\beta|-|\tilde{\alpha}|}e^{\delta\langle\xi\rangle}$$

for any $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in (\mathbb{Z}_+)^n$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\langle\xi\rangle \geq R_0(|\alpha| + |\beta|)$, where $a^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x a(x,\xi)$ and the C_k are independent of α and β. We also write $S^m(R_0, A, B) = S^{m,0}(R_0, A, B)$ and $S^m(R_0, A) = S^m(R_0, A, A)$. We define $S^+(R_0, A, B) := \bigcap_{\delta>0} S^{0,\delta}(R_0, A, B)$.

(ii) Let $R_0 \geq 1$, $m_j, \delta_j \in \mathbb{R}$ ($j = 1, 2$), $A_j \geq 0$ ($j = 1, 2$) and $B \geq 0$, and let $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(\xi, y, \eta) \in S^{m_1,m_2,\delta_1,\delta_2}(R_0, A_1, B, A_2)$ if $a(\xi, y, \eta)$ satisfies

$$\begin{aligned}|\partial^{\alpha+\tilde{\alpha}}_\xi D^{\beta^1+\beta^2+\tilde{\beta}}_y \partial^{\gamma+\tilde{\gamma}}_\eta a(\xi, y, \eta)| &\leq C_{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|}(A_1/R_0)^{|\alpha|}(B/R_0)^{|\beta^1|+|\beta^2|} \\ &\quad \times (A_2/R_0)^{|\gamma|}\langle\xi\rangle^{m_1+|\beta^1|-|\tilde{\alpha}|}\langle\eta\rangle^{m_2+|\beta^2|-|\tilde{\gamma}|}\exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle]\end{aligned}$$

for any $\alpha, \tilde{\alpha}, \beta^1, \beta^2, \tilde{\beta}, \gamma, \tilde{\gamma} \in (\mathbb{Z}_+)^n$, $(\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $\langle\xi\rangle \geq R_0(|\alpha|+|\beta^1|)$ and $\langle\eta\rangle \geq R_0(|\gamma|+|\beta^2|)$. We also write $S^{m_1,m_2,\delta_1,\delta_2}(R_0, A) = S^{m_1,m_2,\delta_1,\delta_2}(R_0, A, A, A)$. Similarly, we define $S^+(R_0, A_1, B, A_2) = \bigcap_{\delta>0} S^{0,0,\delta,\delta}(R_0, A_1, B, A_2)$.

(iii) Let $A, B \geq 0$, and let $a(x,\xi) \in C^\infty(\Gamma)$. We say that $a(x,\xi) \in PS^+(\Gamma; R_0, A, B)$ if $a(x,\xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{(\beta)}(x,\xi)| \leq C_{|\tilde{\alpha}|,\delta}A^{|\alpha|}B^{|\beta|}|\alpha|!|\beta|!\langle\xi\rangle^{-|\alpha|-|\tilde{\alpha}|}e^{\delta\langle\xi\rangle}$$

for any $\alpha, \tilde{\alpha}, \beta \in (\mathbb{Z}_+)^n$, $(x,\xi) \in \Gamma$ with $|\xi| \geq 1$ and $\langle\xi\rangle \geq R_0|\alpha|$ and $\delta > 0$. We also write $PS^+(\Gamma; R_0, A) = PS^+(\Gamma; R_0, A, A)$. Moreover, we say that $a(x,\xi) \in PS^+(X; R_0, A, B)$ if $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$ and $a(x,\xi) \in PS^+(X \times (\mathbb{R}^n \setminus \{0\}); R_0, A, B)$.

(iv) Let $m, \delta \in \mathbb{R}$ and $A, C_0 \geq 0$, and let $\{a_j(x,\xi)\}_{j\in\mathbb{Z}_+} \in \prod_{j\in\mathbb{Z}_+} C^\infty(\Gamma)$. We say that $a(x,\xi) \equiv \{a_j(x,\xi)\}_{j\in\mathbb{Z}_+} \in FS^{m,\delta}(\Gamma; C_0, A)$ if $a(x,\xi)$ satisfies

$$|a^{(\alpha)}_{j(\beta)}(x,\xi)| \leq CC_0^j A^{|\alpha|+|\beta|}j!|\alpha|!|\beta|!\langle\xi\rangle^{m-j-|\alpha|}e^{\delta\langle\xi\rangle}$$

for any $j \in \mathbb{Z}_+$, $\alpha, \beta \in (\mathbb{Z}_+)^n$ and $(x,\xi) \in \Gamma$ with $|\xi| \geq 1$, where C is independent of α, β and j. We define $FS^+(\Gamma; C_0, A) := \bigcap_{\delta>0} FS^{0,\delta}(\Gamma; C_0, A)$. We also write $a(x,\xi) = \sum_{j=0}^\infty a_j(x,\xi)$ formally. Moreover, we write $FS^+(X; C_0, A) = FS^+(X \times (\mathbb{R}^n \setminus \{0\}); C_0, A)$.

(v) For $a(x,\xi) \equiv \sum_{j=0}^\infty a_j(x,\xi) \in FS^+(\Gamma; C_0, A)$ we define the symbol $({}^t a)(x,\xi)$ by

$$({}^t a)(x,\xi) = \sum_{j=0}^\infty b_j(x,\xi), \quad b_j(x,\xi) = \sum_{k+|\alpha|=j}(-1)^{|\alpha|}a^{(\alpha)}_{k(\alpha)}(x,-\xi)/\alpha!.$$

Remark. It is easy to see that $({}^t a)(x,\xi) \in FS^+(\check{\Gamma}; \max\{C_0, 4nA^2\}, 2A)$. Moreover, we have $({}^t({}^t a))(x,\xi) = a(x,\xi)$.

Let Γ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and assume that $a(x,\xi) \in PS^+(\Gamma; R_0, A)$, where $A \geq 0$ and $R_0 \geq 1$. Let Γ_j ($0 \leq j \leq 2$) be open conic subsets of Γ such that $\Gamma_0 \Subset \Gamma_1 \Subset \Gamma_2 \Subset \Gamma$, and write $\Gamma^0 = \Gamma \cap (\mathbb{R}^n \times S^{n-1})$, where $\Gamma_2 \Subset \Gamma$ implies that $\Gamma_2^0 \Subset \Gamma$. It follows from Proposition 2.2.3 of [6] that there are symbols $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))$ ($R \geq 4$) satisfying $0 \leq \Phi^R(\xi, y, \eta) \leq 1$, supp $\Phi^R \subset \mathbb{R}^n \times \Gamma_2$ and $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1$ with $\langle \eta \rangle \geq R$. Put $a^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta) a(y, \eta)$. Then we have $a^R(\xi, y, \eta) \in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), A + C(\Gamma_1, \Gamma_2))$ for $R \geq \max\{4, R_0\}$. Let $u \in \mathcal{C}(\Gamma_0^0)$, and choose $v \in \mathcal{F}_0$ so that $v|_{\Gamma_0^0} = u$. Applying Proposition 1.2 with $a(\xi, y, \eta) = a^R(\eta, y, \xi)$ and noting that $a^R(D_x, y, D_y) = {}^r a(D_x, y, D_y)$, we can see that $a^R(D_x, y, D_y)v$ is well-defined and belongs to $\mathcal{F}_0$ if $R \geq \max\{4, R_0, 2e\sqrt{n}(2A + C(\Gamma_1, \Gamma_2))\}$. Moreover, $a^R(D_x, y, D_y)v$ determines an element $(a^R(D_x, y, D_y)v)|_U \in \mathcal{B}(U)$ and, therefore, an element $\mathrm{sp}((a^R(D_x, y, D_y)v)|_U)|_{\Gamma_0^0}$ $(\equiv (a^R(D_x, y, D_y)v)|_{\Gamma_0^0}) \in \mathcal{C}(\Gamma_0^0)$, where U is a bounded open subset of $\mathbb{R}^n$ satisfying $\Gamma_0^0 \subset U \times S^{n-1}$. It follows from Lemma 2.1 of [7] that $(a^R(D_x, y, D_y)v)|_{\Gamma_0^0}$ does not depend on the choice of $\Phi^R(\xi, y, \eta)$ if $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, B)$ and $R \geq R(A, B, \Gamma_0, \Gamma_1)$, where $R(A, B, \Gamma_0, \Gamma_1) > 0$. From Lemma 2.2 of [7] it follows that for each conic subset Ω of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with $\Omega \Subset \Gamma_0$ there is $R(A, \Omega, \Gamma_0, \Gamma_1, \Gamma_2) > 0$ such that $WF_A(a^R(D_x, y, D_y)w) \cap \Omega = \emptyset$ if $R \geq R(A, \Omega, \Gamma_0, \Gamma_1, \Gamma_2)$, $w \in \mathcal{F}_0$ and $WF_A(w) \cap \Gamma_0 = \emptyset$. Therefore, we can define the operator $a(x, D)$: $\mathcal{C}(\Gamma_0^0) \to \mathcal{C}(\Gamma_0^0)$ by $a(x, D)u = (a^R(D_x, y, D_y)v)|_{\Gamma_0^0}$ for $R \gg 1$, and the operator $a(x, D)$: $\mathcal{C}(\Gamma^0) \to \mathcal{C}(\Gamma^0)$. Moreover, it follows from Lemma 2.2 of [7] that

$$a(x, D)(w|_{\mathcal{U}}) = (a(x, D)w)|_{\mathcal{U}} \quad \text{for } w \in \mathcal{C}(\mathcal{V}),$$

where $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}^n \times S^{n-1}$ satisfying $\mathcal{U} \subset \mathcal{V} \subset \Gamma^0$. So we can define $a(x, D)$: $\mathcal{C}_{\Gamma^0} \to \mathcal{C}_{\Gamma^0}$, which is a sheaf homomorphism. Let X be an open subset of $\mathbb{R}^n$, and assume that $a(x, \xi) \in PS^+(X; R_0, A)$. Similarly, taking $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$, we can define the operator $a(x, D)$: $\mathcal{B}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$ and the operator $a(x, D)$: $\mathcal{B}(U)/\mathcal{A}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$, where U is a bounded open subset of X and $\mathcal{A}(U)$ denotes the space of all real analytic functions defined in U (see, also, §2.7 of [6]). In doing so, we may choose $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C_*)$ so that supp $\Phi^R \subset \mathbb{R}^n \times X_2 \times \mathbb{R}^n$ and $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^n$, where $\Gamma_j = X_j \times (\mathbb{R}^n \setminus \{0\})$. Moreover, we can define the operator $a(x, D)$: $\mathcal{B}_X \to \mathcal{B}_X/\mathcal{A}_X$ and the operator $a(x, D)$: $\mathcal{B}_X/\mathcal{A}_X \to \mathcal{B}_X/\mathcal{A}_X$, which are sheaf homomorphisms.

Here $\mathcal{A}_X$ denotes the sheaf (of germs) of real analytic functions on X.

Assume that $a(x,\xi) \equiv \sum_{j=0}^\infty a_j(x,\xi) \in FS^+(\Gamma; C_0, A)$. Choose $\{\phi_j^R(\xi)\}_{j\in\mathbb{Z}_+} \subset C^\infty(\mathbb{R}^n)$ so that $0 \le \phi_j^R(\xi) \le 1$,

$$\phi_j^R(\xi) = \begin{cases} 0 & \text{if } \langle\xi\rangle \le 2Rj, \\ 1 & \text{if } \langle\xi\rangle \ge 3Rj, \end{cases}$$
$$|\partial_\xi^{\alpha+\beta}\phi_j^R(\xi)| \le \widehat{C}_{|\beta|}(\widehat{C}/R)^{|\alpha|}\langle\xi\rangle^{-|\beta|} \quad \text{if } |\alpha| \le 2j,$$

where the $\widehat{C}_{|\beta|}$ and $\widehat{C}$ do not depend on j and R (see §2.2 of [6]). Then it follows from Lemma 2.2.4 of [6] that

$$\tilde{a}(x,\xi) := \sum_{j=0}^\infty \phi_j^{R/2}(\xi)a_j(x,\xi) \in PS^+(\Gamma; R, 2A + 3\widehat{C}, A)$$

if $R > C_0$. So we can define $a(x,D)u \in \mathcal{C}(\Gamma^0)$ by $a(x,D)u = \tilde{a}(x,D)u$. Indeed, applying the same argument as in §3.7 of [6] we can see that $a(x,D)u \in \mathcal{C}(\Gamma^0)$ does not depend on the choice of $\{\phi_j^R(\xi)\}$. Similarly, $a(x,D)$ defines a sheaf homomorphism $a(x,D)$: $\mathcal{C}_{\Gamma^0} \to \mathcal{C}_{\Gamma^0}$. If $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$, then we can also define the operator $a(x,D)$: $\mathcal{B}(U)/\mathcal{A}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$ and the operator $a(x,D)$: $\mathcal{B}_X/\mathcal{A}_X \to \mathcal{B}_X/\mathcal{A}_X$, where U is an open subset satisfying $U \Subset X$.

Let Γ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let $p(x,\xi) \in FS^+(\Gamma; C_0, A)$, where $A, C_0 \ge 0$.

Theorem 1.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be bounded open subsets of Γ^0 in $\mathbb{R}^n \times S^{n-1}$ such that $\mathcal{V} \Subset \mathcal{U} \Subset \Gamma^0$. Assume that $WF_A(f) \cap \mathcal{U} = \emptyset$ if $f \in L^2(\mathbb{R}^n)$, $WF_A(f) \cap \partial\mathcal{U} = \emptyset$ and $p(x,D)(f|_\mathcal{U}) = 0$ in $\mathcal{C}(\mathcal{U})$. Then $({}^tp)(x,D)$ maps $\mathcal{C}(\check{\mathcal{V}})$ onto $\mathcal{C}(\check{\mathcal{V}})$, i.e., for any $f \in \mathcal{C}(\check{\mathcal{V}})$ there is $u \in \mathcal{C}(\check{\mathcal{V}})$ satisfying $({}^tp)(x,D)u = f$ in $\mathcal{C}(\check{\mathcal{V}})$.*

Corollary 1.5 ([7])**.** *Let $z^0 = (x^0,\xi^0) \in \Gamma$, and assume that $p(x,D)$ is analytic microhypoelliptic at z^0, i.e., there is an open neighborhood $\mathcal{U}$ of $(x^0, \xi^0/|\xi^0|)$ in Γ^0 such that the sheaf homomorphism $p(x,D) : \mathcal{C}_\mathcal{U} \to \mathcal{C}_\mathcal{U}$ is injective. Then $({}^tp)(x,D)$ is microlocally solvable at $(x^0, -\xi^0)$, i.e., there is an open neighborhood $\mathcal{U}$ of $(x^0, \xi^0/|\xi^0|)$ in Γ^0 such that $({}^tp)(x,D) : \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ is surjective.*

Corollary 1.6. *Assume that $p(x,\xi) \equiv \sum_{j=0}^\infty p_j(x,\xi) \in FS^{m,0}(\Gamma; C_0, A)$, and that $p_0(x,\xi)$ is positively homogeneous of degree m in ξ. Let $\mathcal{U}$ and $\mathcal{V}$ be bounded open subsets of Γ^0 satisfying $\mathcal{V} \Subset \mathcal{U}$, and assume that there is a continuous vector field $\vartheta : \mathcal{U} \ni z \mapsto \vartheta(z) \in \mathbb{R}^{2n}$ such that $p_0(x,\xi)$ is microhyperbolic with respect to $\vartheta(z)$*

at each $z \in \mathcal{U}$. Moreover, we assume that for any $z^0 \in \mathcal{U}$ there is no generalized semi-bicharacteristics $\{z(s)\}_{s\in(-\infty,0]}$ of p_0 starting from z^0 in the negative direction such that $(x(s), \xi(s)/|\xi(s)|) \in \mathcal{U}$ for $s \in (-\infty, 0]$, where the parameter s of the curve is chosen so that $-s$ coincides with the arc length from z^0 to $z(s)$ and $z(s) = (x(s), \xi(s))$. For terminology we refer to §4.3 of [6]. *Then $({}^t p)(x, D) : \mathcal{C}(\check{\mathcal{V}}) \to \mathcal{C}(\check{\mathcal{V}})$ is surjective.*

Corollary 1.7. *Let $z^0 = (x^0, \xi^0) \in \Gamma$, and assume that $p(x, \xi) \equiv \sum_{j=0}^{\infty} p_j(x, \xi) \in FS^{m,0}(\Gamma; C_0, A)$, and that $p_0(x, \xi)$ is positively homogeneous of degree m in ξ and microhyperbolic with respect to $(0, e_1) \in \mathbb{R}^{2n}$ at z^0, where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$. Then $({}^t p)(x, D)$ is microlocally solvable at $(x^0, -\xi^0)$.*

Remark. The above corollary was proved in Theorem 5.4.1 of [6] in a different way.

Theorem 1.4 can be proved in the same way as in [7]. We shall give the outline of the proof in the next section. Then Corollary 1.5 easily follows from Theorem 1.4. Combining Theorem 4.3.8 of [6] and Theorem 1.4 one can easily prove Corollary 1.6. Corollary 1.7 is an immediate consequence of Corollary 1.6.

2. Proof of Theorem 1.4

Let Γ_j ($j = 1, 2$) be open conic subsets of Γ such that $\mathcal{V} \Subset \mathcal{U} \Subset \Gamma_1^0 \Subset \Gamma_2^0 \Subset \Gamma^0$, where $\Gamma_j^0 = \Gamma_j \cap (\mathbb{R}^n \times S^{n-1})$. Choose $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))$ ($R \geq 4$) so that $0 \leq \Phi^R(\xi, y, \eta) \leq 1$, supp $\Phi^R \subset \mathbb{R}^n \times \Gamma_2$ and $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1$ with $\langle \eta \rangle \geq R$. We put

$$p^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta) \sum_{j=0}^{\infty} \phi_j^{R/2}(\eta) p_j(y, \eta),$$

where $R > \max\{4, C_0\}$. Then we have

$$p^R(\xi, y, \eta) \in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), 2A + 3\widehat{C} + C(\Gamma_1, \Gamma_2)).$$

By definition there is $R(A, \mathcal{U}, \Gamma_1, \Gamma_2) > \max\{4, C_0\}$ such that

$$(p^R(D_x, y, D_y)v)|_{\mathcal{U}} = p(x, D)(v|_{\mathcal{U}}) \quad \text{in } \mathcal{C}(\mathcal{U}) \tag{2.1}$$

if $R \geq R(A, \mathcal{U}, \Gamma_1, \Gamma_2)$ and $v \in \mathcal{F}_0$. Let Ω_j ($j = 1, 2$) be open conic subset satisfying $\mathcal{V} \Subset \Omega_2^0 \Subset \Omega_1^0 \Subset \mathcal{U}$, and let $\Psi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Omega_2, \Omega_1), C(\Omega_2, \Omega_1))$ ($R \geq$

4) satisfy supp $\Psi^R \subset \mathbb{R}^n \times \Omega_1$ and $\Psi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Omega_2$ with $\langle \eta \rangle \geq R$. We assume that $R \geq \max\{R(A, \mathcal{U}, \Gamma_1, \Gamma_2), 25e\sqrt{n}\max\{2A + C(\Gamma_1, \Gamma_2), C(\Omega_2, \Omega_1)\}\}$. For $\varepsilon, \nu \in \mathbb{R}$ we define

$$L^2_{\varepsilon,\nu} := \{f \in \mathcal{S}'_{-\varepsilon};\ \langle x \rangle^\nu e^{\varepsilon\langle D \rangle} f(x) \in L^2(\mathbb{R}^n)\}.$$

$L^2_{\varepsilon,\nu}$ is a Hilbert space in which the scalar product is given by

$$(f, g)_{L^2_{\varepsilon,\nu}} := (\langle x \rangle^\nu e^{\varepsilon\langle D \rangle} f, \langle x \rangle^\nu e^{\varepsilon\langle D \rangle} g)_{L^2},$$

where $(\cdot, \cdot)_{L^2}$ denotes the scalar product of $L^2(\mathbb{R}^n)$. We denote by $\mathcal{X}$ the inductive limit $\varinjlim L^2_{1/j,1/j}$ of the sequence $\{L^2_{1/j,1/j}\}$ (as a locally convex space). Define an operator $T : L^2(\mathbb{R}^n) \to \mathcal{X} \times \mathcal{X}$ as follows;

(i) the domain $D(T)$ of T is given by

$$D(T) = \{f \in L^2(\mathbb{R}^n);\ (1 - \Psi^R(D_x, y, D_y))f \in \mathcal{X} \text{ and } p^R(D_x, y, D_y)f \in \mathcal{X}\},$$

(ii) $Tf = ((1 - \Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f)$ for $f \in D(T)$.

Let $f \in D(T)$. Then (2.1) gives $p(x, D)(f|_{\mathcal{U}}) = 0$ in $\mathcal{C}(\mathcal{U})$. Moreover, it follows from Lemma 2.1 of [7] that there is $R(\Omega_1, \Omega_2, \mathcal{U}) > 0$ such that $WF_A(f) \cap \partial\mathcal{U} = \emptyset$ if $R \geq R(\Omega_1, \Omega_2, \mathcal{U})$. Therefore, by the assumption of Theorem 1.4 we have $WF_A(f) \cap \mathcal{U} = \emptyset$. From Lemma 2.9 of [7] there are $R_1(\Omega_1, \Omega_2, \mathcal{U}) > 0$ and $\delta(f, \Omega_1, \mathcal{U}) > 0$ such that $\Psi^R(D_x, y, D_y)f \in L^2_{\delta,\nu}$ if $R \geq R_1(\Omega_1, \Omega_2, \mathcal{U})$, $\nu \in \mathbb{R}$ and $\delta < \min\{1/(2R), \delta(f, \Omega_1, \mathcal{U})\}$. This implies that $f \in \mathcal{X}$, *i.e.*, $D(T) = \mathcal{X}$. We can easily prove that T is a closed operator (see §3 of [7]).

Repeating the same argument as in §3 of [7], we can show that for any $f \in \mathcal{A}'(\mathbb{R}^n)$ there is $u \in \mathcal{F}_0$ satisfying

$$({}^t p)(x, D)(u|_{\check{\mathcal{V}}}) = f|_{\check{\mathcal{V}}} \quad \text{in } \mathcal{C}(\check{\mathcal{V}}),$$

which proves Theorem 1.4.

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