Remarks on the conditions (L) and $(\mathrm{L})_0$ in the paper "On the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time dependent coefficients"

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Let the conditions (L) and $(\mathrm{L})_0$ be given in [3]. In [3] we asserted the following lemma.

Lemma 1. *The condition* $(\mathrm{L})_0$ *is satisfied if the condition* (L) *is satisfied.*

In this note we shall prove Lemma 1. Let U be an open subset of $\mathbf{R}^n$, and let $a(t,\xi)$ be a real analytic function defined in $[0,\delta_0]\times\overline{U}$, where $\delta_0>0$. Then there is a compact complex neighborhood Ω_a of $[0,\delta_0]$ such that $a(t,\xi)$ is regarded as an analytic function defined in Ω_a for $\xi\in\overline{U}$. We assume that $a(t,\xi)\ge 0$ for $(t,\xi)\in[0,\delta_0]\times\overline{U}$. Let $b(t,\xi)$ be real analytic in $[0,\delta_0]\times\overline{U}$. Let $\mathcal{R}_U(\xi): U\ni\xi\mapsto\mathcal{R}_U(\xi)\in\mathcal{P}(\mathbf{C})$ satisfy $\#\mathcal{R}_U(\xi)\le N_U$ for any $\xi\in U$, where $N_U\in\mathbf{N}$ and $\#A$ denotes the number of the elements of a set A. We choose $\delta\in(0,1]$ so that $[-\delta,\delta_0+\delta]\subset\Omega_a$. Let $c\in(0,1]$, and let $\mathcal{R}_{a,\delta,c}(\xi)$ $(\subset\mathbf{C})$ be a set-valued function defined for $\xi\in U$ satisfying the following:

(i) $\sup_{\xi\in U}\#\mathcal{R}_{a,\delta,c}(\xi)<\infty$.

(ii) If $\xi\in U$, $a(t,\xi)\not\equiv 0$ in t, $\lambda\in\Omega_a$, $a(\lambda,\xi)=0$, $|\operatorname{Im}\lambda|\le\delta$ and $\operatorname{Re}\lambda\in[-\delta,\delta_0+\delta]$, then there is $s\in\mathcal{R}_{a,\delta,c}(\xi)$ satisfying $|\operatorname{Im}\lambda|\ge c|(\operatorname{Re}\lambda)_+-s|$.

Lemma 1 easily follows from Lemma 2 below and the compactness argument.

Lemma 2. *There are positive constants* δ_1 *and* $A\equiv A(a,\delta,c)$ *such that*

$$(\mathrm{L})_{a,\delta,c}\qquad \min\Big\{\min_{s\in\mathcal{R}_{a,\delta,c}(\xi)}|t-s|,1\Big\}|b(t,\xi)|\le AC\sqrt{a(t,\xi)}$$

$$\textit{for } (t,\xi)\in[0,\delta_1]\times U$$

if, with $C \geq 1$*,*

$$(\mathrm{L})_U \quad \min\Big\{\min_{s\in\mathcal{R}_U(\xi)} |t-s|, 1\Big\} |b(t,\xi)| \leq C\sqrt{a(t,\xi)} \quad \textit{for } (t,\xi) \in [0,\delta_1]\times U,$$

where $\min_{s\in\emptyset} |t-s| = 1$.

Proof. Assume that $(\mathrm{L})_U$ holds, and put

$$\kappa(\xi) = \int_0^{\delta_0} a(t,\xi)\, dt.$$

If $\kappa(\xi) \equiv 0$, then $a(t,\xi) \equiv b(t,\xi) \equiv 0$ in (t,ξ) and the lemma becomes trivial. Assume that $\kappa(\xi) \not\equiv 0$. Let $\xi^0 \in \overline{U}$. We apply Hironaka's resolution theorem to $\kappa(\xi)$ (see [1]). Then there are an open neighborhood $U(\xi^0)$ of ξ^0, a real analytic manifold $\widetilde{U}(\xi^0)$, a proper analytic mapping $\varphi \equiv \varphi(\xi^0)$: $\widetilde{U}(\xi^0) \ni \tilde{u} \mapsto \varphi(\tilde{u}) (\equiv \varphi(\tilde{u};\xi^0)) \in U(\xi^0)$ satisfying the following:

(i) φ: $\widetilde{U}(\xi^0) \setminus \widetilde{A} \to U(\xi^0) \setminus A$ is an isomorphism, where $A = \{\xi \in \overline{U};\ \kappa(\xi) = 0\}$ and $\widetilde{A} = \varphi^{-1}(A)$.

(ii) For each $p \in \widetilde{U}(\xi^0)$ there are local analytic coordinates $X\,(\equiv X^p) = (X_1,\cdots,X_n)\,(= (X_1^p,\cdots,X_n^p))$ centered at p, $r(p) \in \mathbf{Z}_+$ with $r(p) \leq n$, $s_k(p) \in \mathbf{N}$ ($1 \leq k \leq r(p)$), a neighborhood $\widetilde{U}(\xi^0;p)$ of p and a real analytic function $e(X)\,(\equiv e(X^p;p))$ in $\widetilde{V}(\xi^0;p)$ such that $e(X) > 0$ for $X \in \widetilde{V}(\xi^0;p)$ and

$$\kappa(\varphi(\tilde{u})) = e(X(\tilde{u})) \prod_{k=1}^{r(p)} X_k(\tilde{u})^{2s_k(p)} \quad (\ \tilde{u} \in \widetilde{U}(\xi^0;p)),$$

where $\widetilde{V}(\xi^0;p) = \{X(\tilde{u});\ \tilde{u} \in \widetilde{U}(\xi^0;p)\}$ and $\prod_{k=1}^{r(p)} \cdots = 1$ if $r(p) = 0$.

Here $\widetilde{V}(\xi^0;p)$ is a neighborhood of 0 in $\mathbf{R}^n$. Define $\tilde{\varphi}\,(\equiv \tilde{\varphi}(\xi^0,p)) : \widetilde{V}(\xi^0;p) \to U(\xi^0)$ by $\tilde{\varphi}(X(\tilde{u}))\,(\equiv \tilde{\varphi}(X^p(\tilde{u});\xi^0,p)) = \varphi(\tilde{u})\,(\equiv \varphi(\tilde{u};\xi^0))$ for $\tilde{u} \in \widetilde{U}(\xi^0;p)$. Let $U_0(\xi^0)$ be a compact neighborhood of ξ^0 in $U(\xi^0)$, and put $\widetilde{U}_0(\xi^0) = \varphi^{-1}(U_0(\xi^0))$. Fix $p \in \widetilde{U}_0(\xi^0)$, and put

$$\alpha(p) = (s_1(p),\cdots,s_{r(p)}(p),0,\cdots,0) \in (\mathbf{Z}_+)^n.$$

From $(\mathrm{L})_U$ it is easy to see that there is a real analytic function $d(t,X;p)$ defined in $[0,\delta_0] \times \widetilde{V}(\xi^0;p)$ satisfying

$$|b(t,\tilde{\varphi}(X;\xi^0,p))|^2 = d(t,X;p)X^{2\alpha(p)} \quad \text{for } (t,X) \in [0,\delta_0] \times \widetilde{V}(\xi^0;p).$$

From (2.6) and (2.7) of [3] we can also write

$$a(t, \tilde{\varphi}(X; \xi^0, p)) = c(t, X; p) f(t, X; p) X^{2\alpha(p)},$$
$$f(t, X; p) = t^{m(p)} + a_1(X; p) t^{m(p)-1} + \cdots + a_{m(p)}(X; p)$$

for $(t, X) \in [0, \delta(p)] \times \widetilde{V}(p)$, where $0 < \delta(p) \leq \min\{\delta_0, 1\}$, $\widetilde{V}(p)$ is a compact neighborhood of 0 in $\widetilde{V}(\xi^0; p)$, $m(p) \in \mathbf{Z}_+$, $c(t, X; p)$ is a real analytic function defined in $[0, \delta(p)] \times \widetilde{V}(p)$ satisfying $c(t, X; p) > 0$ and the $a_k(X; p)$ are real analytic functions defined in $\widetilde{V}(p)$. By the Weierstrass division theorem there are a polynomial $g(t, X; p)$ of t with real analytic coefficients of X defined in $\widetilde{V}(p)$, and a real analytic function $h(t, X; p)$ defined in $[0, \delta(p)] \times \widetilde{V}(p)$ satisfying $\deg_t g(t, X; p) < m(p)$ and

$$d(t, X; p) = h(t, X; p) f(t, X; p) + g(t, X; p) \quad \text{in } [0, \delta(p)] \times \widetilde{V}(p),$$

modifying $\delta(p)$ and $\widetilde{V}(p)$ if necessary. Fix $X \in \widetilde{V}(p)$ with $X^{\alpha(p)} \neq 0$, and put $\xi = \tilde{\varphi}(X; \xi^0, p)$. Then $(\mathrm{L})_U$ implies that

$$\text{(1)} \qquad \min\Big\{ \min_{s \in \mathcal{R}_U(\xi)} |t - s|^2, 1 \Big\} |g(t, X; p)| \leq C^2 C_1(p) f(t, X; p)$$

for $t \in [0, \delta(p)]$, where

$$C_1(p) = \max_{(s, Y) \in [0, \delta(p)] \times \widetilde{V}(p)} (c(s, Y; p) + |h(s, Y; p)|) + 1.$$

Let us prove that there is a positive constant $A(p, \delta, c)$ independent of X such that

$$\text{(2)} \qquad \min\Big\{ \min_{s \in \mathcal{R}_{a,\delta,c}(\xi)} |t - s|^2, 1 \Big\} |g(t, X; p)| \leq A(p, \delta, c) C^2 C_1(p) f(t, X; p)$$

for $t \in [0, \delta(p)]$. If $t \in [0, \delta(p)] \cap \mathcal{R}_{a,\delta,c}(\xi)$, then (2) holds. So we may assume that $t \in [0, \delta(p)] \setminus \mathcal{R}_{a,\delta,c}(\xi)$ and that $g(s, X; p) \not\equiv 0$ in s. If $\mathcal{R}_{a,\delta,c}(\xi) = \emptyset$, then we have $a(s, \xi) \not\equiv 0$ in s and

$$a(\lambda, \xi) \neq 0 \quad \text{if } \lambda \in \Omega_a,\ \mathrm{Re}\,\lambda \in [-\delta, \delta_0 + \delta] \text{ and } |\mathrm{Im}\,\lambda| \leq \delta,$$

since $\kappa(\xi) \neq 0$. This implies that

$$f(t, X; p) \geq \delta^{m(p)} \quad \text{if } \mathcal{R}_{a,\delta,c}(\xi) = \emptyset.$$

Therefore, (2) is valid if $\mathcal{R}_{a,\delta,c}(\xi) = \emptyset$ and

$$A(p, \delta, c) \geq \delta^{-m(p)} \max_{(s, Y) \in [0, \delta(p)] \times \widetilde{V}(p)} |g(s, Y; p)|.$$

Thus we may assume that $\mathcal{R}_{a,\delta,c}(\xi) \neq \emptyset$. Then there is $\lambda_0 \in \mathcal{R}_{a,\delta,c}(\xi)$ satisfying $\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |t-\lambda| = |t-\lambda_0|$. Now let us apply the argument in the proof of Lemma 2.1 of [2]. First consider the case where $t \geq \delta(p)/2$. We put

$$s_0 = \max\{0, t - |t-\lambda_0|\},$$

and divide the interval $(s_0, t]$ into $(m(p)+N_U)$ subintervals with equal length. Write

$$g(s, X; p) = d(p) \prod_{j=1}^{\bar{\mu}} (s - \mu_j),$$

where $\bar{\mu} < m(p)$ and $\mu_j \in \mathbf{C}$. Then there is $\hat{k} \in \mathbf{N}$ such that $\hat{k} \leq m(p) + N_U$ and

$$(s_0 + (\hat{k}-1)\rho, s_0 + \hat{k}\rho] \cap (\{\operatorname{Re}\mu_j;\ 1 \leq j \leq \bar{\mu}\} \cup \{\operatorname{Re}\lambda;\ \lambda \in \mathcal{R}_U(\xi)\}) = \emptyset,$$

where $\rho = (t - s_0)/(m(p) + N_U)$. Put

$$I = [s_0 + (\hat{k} - 2/3)\rho, s_0 + (\hat{k} - 1/3)\rho].$$

(i) Let $\tilde{t} \in I$. Then we have

$$\min\{\min_{\lambda \in \mathcal{R}_U(\xi)} |\tilde{t} - \lambda|, 1\} \geq \rho/3, \tag{3}$$

$$\min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|, 1\} \leq \min\{|\tilde{t} - \lambda_0|, 1\} \leq 2\min\{|t - \lambda_0|, 1\}, \tag{4}$$

since $\rho \leq 1$ and $0 < t - s_0 \leq |t - \lambda_0|$. Since $\rho = |t-\lambda_0|/(m(p)+N_U)$ if $t \geq |t-\lambda_0|$, and $\rho = t/(m(p)+N_U)$ otherwise, we have

$$\rho \geq \delta(p)\min\{|t-\lambda_0|, 1\}/(2(m(p)+N_U)). \tag{5}$$

This, together with (3) and (4), gives

$$\begin{aligned} \min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|, 1\} &\leq 4(m(p)+N_U)\rho/\delta(p) \\ &\leq 12(m(p)+N_U)\min\{\min_{\lambda \in \mathcal{R}_U} |\tilde{t}-\lambda|, 1\}/\delta(p). \end{aligned} \tag{6}$$

From (1) and (6) we have

$$\begin{aligned} &\min\{\min_{\lambda \in \mathcal{R}_{a,\delta,c}(\xi)} |\tilde{t} - \lambda|^2, 1\} |g(\tilde{t}, X; p)| \\ &\leq 144(m(p)+N_U)^2 C^2 C_1(p) f(\tilde{t}, X; p)/\delta(p)^2. \end{aligned} \tag{7}$$

(ii) (5) implies that

$$\min\{\min_{\lambda\in\mathcal{R}_{a,\delta,c}(\xi)}|t-\lambda|,1\}=\min\{|t-\lambda_0|,1\}\le 2(m(p)+N_U)\rho/\delta(p).$$

On the other hand, we have

$$\min\{\min_{\lambda\in\mathcal{R}_{a,\delta,c}(\xi)}|\tilde t-\lambda|,1\}\ge\rho/3\quad\text{for }\tilde t\in I,$$

since $|\tilde t-\lambda|\ge|t-\lambda|-|t-\tilde t|\ge|t-\lambda_0|-|t-\tilde t|$ for $\lambda\in\mathcal{R}_{a,\delta,c}(\xi)$, and $|t-\tilde t|=t-\tilde t\le t-s_0-\rho/3\le|t-\lambda_0|-\rho/3$ for $\tilde t\in I$, This, together with (5), gives

$$\begin{aligned}(8)\quad &\min\{\min_{\lambda\in\mathcal{R}_{a,\delta,c}(\xi)}|t-\lambda|,1\}=\min\{|t-\lambda_0|,1\}\\ &\le 2(m(p)+N_U)\rho/\delta(p)\le 6(m(p)+N_U)\min\{\min_{\lambda\in\mathcal{R}_{a,\delta,c}(\xi)}|\tilde t-\lambda|,1\}/\delta(p)\end{aligned}$$

for $\tilde t\in I$. It is obvious that $|\tilde t-\operatorname{Re}\mu_j|\ge\rho/3$ and

$$\begin{aligned}&|t-\operatorname{Re}\mu_j|\le|\tilde t-\operatorname{Re}\mu_j|\quad\text{if }(t+\tilde t)/2\le\operatorname{Re}\mu_j,\\ &|t-\operatorname{Re}\mu_j|\le 2|\tilde t-\operatorname{Re}\mu_j|\quad\text{if }\operatorname{Re}\mu_j\le 2\tilde t-t,\\ &0<t-\operatorname{Re}\mu_j\le 2(t-\tilde t)\le 2(m(p)+N_U-1/3)\rho\\ &\qquad\qquad\text{if }2\tilde t-t\le\operatorname{Re}\mu_j\le(t+\tilde t)/2,\end{aligned}$$

for $\tilde t\in I$ and $1\le j\le\bar\mu$. Noting that $|\tilde t-\operatorname{Re}\mu_j|\ge\rho/3$, we have

$$|t-\operatorname{Re}\mu_j|\le 6(m(p)+N_U)|\tilde t-\operatorname{Re}\mu_j|\quad\text{for }\tilde t\in I\text{ and }1\le j\le\bar\mu,$$

which gives

$$(9)\qquad |g(t,X;p)|\le\{6(m(p)+N_U)\}^{m(p)-1}|g(\tilde t,X;p)|\quad\text{for }\tilde t\in I.$$

We write

$$f(s,X;p)=\prod_{j=1}^{m(p)}(s-\lambda_j).$$

We may assume that $f(s,X;p)$ is defined in $\mathbf{R}\times\widetilde V(p)$, $\operatorname{Re}\lambda_j\in[-\delta(p),\delta(p)]$ for $1\le j\le m(p)$, modifying $\widetilde V(p)$ if necessary. Let $1\le j\le m(p)$. If $|\operatorname{Im}\lambda_j|>\delta$, then we have

$$(10)\quad |\tilde t-\lambda_j|^2\le 4\delta(p)^2+|\operatorname{Im}\lambda_j|^2\le(1+(2/\delta)^2)|\operatorname{Im}\lambda_j|^2\le(3/\delta)^2|\operatorname{Im}\lambda_j|^2$$

for $\tilde{t} \in I$. If $|\operatorname{Im}\lambda_j| \le \delta$ and $\operatorname{Re}\lambda_j < -\delta$, then we have

$$(11)\quad |\tilde{t}-\lambda_j|^2 \le 4\delta(p)^2+\delta^2 \le (1+(2/\delta)^2)|t-\lambda_j|^2 \le (3/\delta)^2|t-\lambda_j|^2 \quad \text{for } \tilde{t} \in I.$$

Let $|\operatorname{Im}\lambda_j| \le \delta$ and $\operatorname{Re}\lambda_j \ge -\delta$. Then there is $s_j \in \mathcal{R}_{a,\delta,c}(\xi)$ satisfying $|\operatorname{Im}\lambda_j| \ge c|(\operatorname{Re}\lambda_j)_+ - s_j|$. Therefore, we have

$$\begin{aligned}(12)\qquad |t-\lambda_j| &\ge (c|t-(\operatorname{Re}\lambda_j)_+| + |\operatorname{Im}\lambda_j|)/2\\ &\ge (c|t-s_j| - c|(\operatorname{Re}\lambda_j)_+ - s_j| + |\operatorname{Im}\lambda_j|)/2\\ &\ge c|t-s_j|/2 \ge c|t-\lambda_0|/2 \ge c|\tilde{t}-\lambda_j|/4 \quad \text{if } |\tilde{t}-\lambda_j| \le 2|t-\lambda_0|.\\ (13)\qquad |t-\lambda_j| &\ge |\tilde{t}-\lambda_j| - |t-\tilde{t}| \ge |\tilde{t}-\lambda_j| - |t-s_0| \ge |\tilde{t}-\lambda_j|/2\\ &\qquad\qquad \text{if } |\tilde{t}-\lambda_j| \ge 2|t-\lambda_0|.\end{aligned}$$

It follows from (10) – (13) that

$$(14)\qquad f(t,X;p) \ge (\min\{\delta/3, c/4\})^{m(p)} f(\tilde{t},X;p) \quad \text{for } \tilde{t} \in I.$$

(iii) From (7) – (9) and (14) we have

$$\begin{aligned}&\min\{\min_{\lambda\in\mathcal{R}_{a,\delta,c}(\xi)} |t-\lambda|^2, 1\}|g(t,X;p)|\\ &\le 4\cdot 6^{3+m(p)}(m(p)+N_U)^{3+m(p)}(\min\{\delta/3,c/4\})^{-m(p)}\delta(p)^{-4}\\ &\quad\times C^2C_1(p)f(t,X;p).\end{aligned}$$

So (2) holds if $\mathcal{R}_{a,\delta,c}(\xi) \ne \emptyset$, $t \in [\delta(p)/2, \delta(p)]$ and

$$A(p,\delta,c) \ge 4\cdot 6^{3+m(p)}(m(p)+N_U)^{3+m(p)}(\min\{\delta/3,c/4\})^{-m(p)}\delta(p)^{-4}.$$

Assume that $t < \delta(p)/2$. Then, putting

$$s_0 = \min\{\delta(p), t+|t-\lambda_0|\}$$

and dividing the interval $[t, \delta_0)$ into $(m(p)+N_U)$ subintervals with equal length we repeat the arguments above to prove (2). (2) yields

$$\min\{\min_{s\in\mathcal{R}_{a,\delta,c}(\xi)} |t-s|^2, 1\}|b(t,\xi)|^2 \le A'(p,\delta,c)C^2C_1(p)a(t,\xi)$$
$$\text{for } t\in[0,\delta(p)] \text{ and } \xi = \tilde{\varphi}(X,\xi^0,p) \text{ with } X \in \widetilde{V}(p),$$

where

$$A'(p,\delta,c) = A(p,\delta,c)\max_{(s,Y)\in[0,\delta(p)]\times\widetilde{V}(p)} |c(s,Y;p)|^{-1}(1+|h(s,Y;p)|).$$

Put $\widetilde{U}(p) = (X^p)^{-1}(\widetilde{V}(p))\,(\subset \widetilde{U}(\xi^0;p))$. Since $\overline{U}$ is compact, there are $N \in \mathbf{N}$ and $\xi^j \in \overline{U}$ ($1 \le j \le N$) such that $\overline{U} \subset \bigcup_{j=1}^N \mathring{U}_0(\xi^j)$. Here $\mathring{A}$ denotes the interior of A ($\subset \mathbf{R}^n$). Since $\widetilde{U}_0(\xi^j)$ is compact, there are $P_j \in \mathbf{N}$ and $p^{j,k} \in \widetilde{U}_0(\xi^j)$ ($1 \le k \le P_j$) such that $\widetilde{U}_0(\xi^j) \subset \bigcup_{k=1}^{P_j} \widetilde{U}(p^{j,k})$. Therefore, putting

$$A(a,\delta,c) = \max\{A'(p^{j,k},\delta,c)^{1/2}C_1(p^{j,k})^{1/2};\ 1 \le j \le N \text{ and } 1 \le k \le P_j\},$$

we complete the proof of Lemma 2. □

References

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