Remarks on the composition formula for classical pseudo-differential operators

Seiichiro Wakabayashi

1. Introduction

The composition formula was given in §18.5 of Hörmander [1] for classical pseudo-differential operators whose symbols belong to symbol classes defined by Hörmander metrics and Hörmander weights. The formula was proved via results of the Weyl calculus. However, there is a loss in doing so (see Example 8 at the end of §2). In this note we will give the composition formula for classical pseudo-differential operators, applying directly the arguments in §18.4 of [1]. Another aim of this note is to make the proofs in §18.4 of [1] clearly understandable.

Let g_j ($j = 1,2$) be σ temperate Riemannian metrics in $\mathbb{R}^{2n}$. Then the g_j satisfy the following:

(i) The g_j are slowly varying, *i.e.*, there are positive constants $c(g_j)$ and $C_0(g_j)$ ($j = 1,2$) such that

(1) $$g_{jX+Y}(t) \le C_0(g_j) g_{jX}(t) \quad \text{for any } t \in \mathbf{R}^{2n} \quad \text{if } X,Y \in \mathbb{R}^{2n} \text{ and } g_{jX}(Y) \le c(g_j).$$

(ii) There are $C_1(g_j) > 0$ and $N(g_j)$ such that

$$g_{jX}(t) \le C_1(g_j) g_{jY}(t)(1 + g^{\sigma}_{jX}(X-Y))^{N(g_j)} \quad \text{for } X,Y,t \in \mathbb{R}^{2n},$$

where

$$g^{\sigma}_{jX}(Y) = \sup_{t \in \mathbb{R}^{2n} \setminus \{0\}} \sigma(Y,t)^2 / g_{jX}(t),$$
$$\sigma((x,\xi),(y,\eta)) = \langle y,\xi\rangle - \langle x,\eta\rangle,$$
$$\langle y,\xi\rangle = \sum_{j=1}^{n} y_j \xi_j \quad \text{for } y = (y_1,\cdots,y_n) \text{ and } \xi = (\xi_1,\cdots.\xi_n).$$

Let $m_j(X)$ ($j = 1,2$) be σ, g_j temperate weights, *i.e.*, let $m_j(X)$ ($j = 1,2$) satisfy the following:

(i) The $m_j(X)$ are g_j continuous, *i.e.*, there are positive constants $c(m_j)$ and $C(m_j)$ ($j=1,2$) such that

$$C(m_j)^{-1} \leq m_j(X+Y)/m_j(X) \leq C(m_j) \tag{2}$$

if $X,Y \in \mathbb{R}^{2n}$ and $g_{jX}(Y) \leq c(m_j)$.

(ii) There are $C_1(m_j) > 0$ and $N(m_j)$ such that

$$m_j(X) \leq C_1(m_j) m_j(Y)(1+g^\sigma_{jX}(X-Y))^{N(m_j)} \quad \text{for } X,Y \in \mathbb{R}^{2n}.$$

Put

$$g = (g_1+g_2)/2.$$

Then g is slowly varying. We assume that the following conditions (A-1) – (A-3) are satisfied:

(A-1) There are $C(g_1,g_2) > 0$ and $N(g_1,g_2)$ such that

$$\begin{aligned} &g^\sigma_{1X}(t) \leq C(g_1,g_2) g^\sigma_{1Y}(t)(1+g^\sigma_{2X}(X-Y))^{N(g_1,g_2)}, \\ &g^\sigma_{2X}(t) \leq C(g_1,g_2) g^\sigma_{2Y}(t)(1+g^\sigma_{1X}(X-Y))^{N(g_1,g_2)} \quad \text{for } X,Y,t \in \mathbb{R}^{2n}. \end{aligned}$$

(A-2) There are $C > 0$ and N such that

$$\begin{aligned} &m_1(X) \leq C m_1(Y)(1+g^\sigma_{2Y}(X-Y))^N, \\ &m_2(X) \leq C m_2(Y)(1+g^\sigma_{1Y}(X-Y))^N \quad \text{for } X,Y \in \mathbb{R}^{2n}. \end{aligned}$$

(A-3) There is $c > 0$ such that

$$g_{1X}(x,\xi) \geq c g_{1X}(0,\xi), \quad g_{2X}(x,\xi) \geq c g_{2X}(x,0) \quad \text{for } X,(x,\xi) \in \mathbb{R}^{2n}.$$

By (A-1) g is σ temperate (see Lemma 4 below). Moreover, the m_j are σ, g temperate (see Lemma 4 below). Define

$$\begin{aligned} &m(x) = m_1(X)m_2(X), \quad M(X,Y) = m_1(X)m_2(Y), \\ &G_{(X,Y)}(s,t) = g_{1X}(s) + g_{2Y}(t), \\ &g_{0\,(X,Y)}(y,\xi) = G_{(X,Y)}(0,\xi,y,0)\,(= g_{1X}(0,\xi)+g_{2Y}(y,0)) \end{aligned}$$

for $X,Y,s,t,(y,\xi) \in \mathbb{R}^{2n}$. Then $m(X)$ is σ, g temperate and G is a slowly varying metric on $\mathbb{R}^{4n}$, *i.e.*, there are positive constants c_0 and C_0 such that

$$C_0^{-1} \leq G_{(X+X_1,Y+Y_1)} \leq C_0 G_{(X,Y)} \tag{3}$$

if $X,Y,X_1,Y_1 \in \mathbb{R}^{2n}$ and $G_{(X,Y)}(X_1,Y_1) \leq c_0$. We may assume that

$$c_0 \leq c(m_j) \quad (\,j=1,2).$$

Let $a_j(x,\xi) \in S(m_j,g_j)$ ($j=1,2$), *i.e.*,

$$\|a_j\|_k \equiv \sup_{X\in\mathbb{R}^{2n}} |a_j|_k^{g_j}(X)/m_j(X) < \infty \quad \text{for any } k \in \mathbb{Z}_+ (\equiv \mathbb{N}\cup\{0\}),$$

where for $u \in C^\infty(\mathbb{R}^{2n})$

$$|u|_k^{g_j}(X) = \sup_{t_1,\cdots,t_k\in\mathbb{R}^{2n}} |u^{(k)}(X;t_1,\cdots,t_k)|/\prod_{l=1}^k g_{jX}(t_l)^{1/2},$$
$$u^{(k)}(X;t_1,\cdots,t_k) = (\partial^k/\partial s_1\cdots\partial s_k)u(X+s_1t_1+\cdots+s_kt_k)|_{s_1=\cdots=s_k=0}.$$

Note that $S(m_j,g_j)$ is a Frechét space with a family of semi-norms $\{\|\cdot\|_k\}_{k=0,1,2,\cdots}$. We also note that $|\cdot|_k^{g_j}$ is invariant under a linear change of coordinate systems in $\mathbb{R}^{2n}$ (the choice of a basis of $\mathbb{R}^{2n}$). Put

$$b(x,\xi) = a_1(x,\xi)\circ a_2(x,\xi),$$

where $a_1(x,\xi)\circ a_2(x,\xi) = \sigma(a_1(x,D)a_2(x,D))$ and $\sigma(a(x,D)) = a(x,\xi)$. If $a_j \in \mathscr{S}(\mathbb{R}^{2n})$ ($j=1,2$), then

(4)
$$b(x,\xi) = (2\pi)^{-n}\int e^{-i\langle y,\eta\rangle} a_1(x,\xi+\eta)a_2(x+y,\xi)\,dyd\eta$$
$$= e^{iA(D_\xi,D_y)}(a_1(x,\xi)a_2(y,\eta))|_{y=x,\,\eta=\xi},$$

where $A(D_\xi,D_y) = \sum_{j=1}^n D_{y_j}D_{\xi_j}$, $D_{y_j} = -i\partial/\partial y_j$ and $D_{\xi_j} = -i\partial/\partial\xi_j$. Indeed, if $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\chi(0)=1$, then

$$\int \hat{f}(\tilde{y},\tilde{\eta})\mathscr{F}^{-1}_{(y,\eta)}[e^{-i\langle y,\eta\rangle}\chi(\varepsilon y)\chi(\varepsilon\eta)](\tilde{y},\tilde{\eta})\,d\tilde{y}d\tilde{\eta}$$
$$= \int f(y,\eta)e^{-i\langle y,\eta\rangle}\chi(\varepsilon y)\chi(\varepsilon\eta)\,dyd\eta \to \int f(y,\eta))e^{-i\langle y,\eta\rangle}\,dyd\eta \quad \text{as } \varepsilon\downarrow 0$$

for $f \in \mathscr{S}(\mathbb{R}^{2n})$, where $\hat{f}(\tilde{y},\tilde{\eta})$ denotes the Fourier transform of f and $\mathscr{F}^{-1}_{(y,\eta)}[f(y,\eta)](\tilde{y},\tilde{\eta})$ denotes the inverse Fourier transform of f. On the other hand, we have

$$\mathscr{F}^{-1}_{(y,\eta)}[e^{-i\langle y,\eta\rangle}\chi(\varepsilon y)\chi(\varepsilon\eta)](\tilde{y},\tilde{\eta}) \to (2\pi)^{-n}e^{i\langle\tilde{y},\tilde{\eta}\rangle} \quad \text{as } \varepsilon\downarrow 0.$$

Since there is $C(\chi)>0$ such that

$$|\hat{f}(\tilde{y},\tilde{\eta})\mathscr{F}^{-1}_{(y,\eta)}[e^{-i\langle y,\eta\rangle}\chi(\varepsilon y)\chi(\varepsilon\eta)](\tilde{y},\tilde{\eta})| \leq C(\chi)|\hat{f}(\tilde{y},\tilde{\eta})|,$$

Lebesgue's convergence theorem gives

$$\int e^{-i\langle y,\eta\rangle} f(y,\eta)\,dyd\eta = (2\pi)^{-n}\int e^{i\langle \tilde{y},\tilde{\eta}\rangle}\hat{f}(\tilde{y},\tilde{\eta})\,d\tilde{y}d\tilde{\eta} \quad \text{for } f\in\mathscr{S}(\mathbb{R}^{2n}).$$

So we have the second equality of (4). Define a $2n\times 2n$ matrix A by

$$A=\frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0\end{pmatrix},$$

where I_n denotes the $n\times n$ identity matrix. Noting that

$$A\begin{pmatrix}\tilde{y}\\ \tilde{\xi}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\tilde{\xi}\\ \tilde{y}\end{pmatrix},$$

we define

$$g^{A}_{0(X,Y)}(y,\xi)=\sup_{g_{0(X,Y)}(A\widetilde{X})<1}\langle\widetilde{X},(y,\xi)\rangle^2$$
$$\Big(=\sup_{g_{0(X,Y)}(\tilde{\xi}/2,\tilde{y}/2)<1}\langle(\tilde{y},\tilde{\xi}),(y,\xi)\rangle^2\Big)\quad\text{for } X,Y,(y,\xi)\in\mathbb{R}^{2n}.$$

It is easy to see that

$$g^{A}_{0(X,Y)}(y,\xi)=4g^{\sigma}_{0(X,Y)}(y,\xi)\Big(=4\sup_{t\in\mathbb{R}^{2n}\setminus\{0\}}\sigma((y,\xi),t)^2/g_{0(X,Y)}(t)\Big).$$

We assume that

(A-4) $g_{0(X,X)}\le g^{A}_{0(X,X)}$ for any $X\in\mathbb{R}^{2n}$.

Put for $l\in\mathbb{Z}_+$ and $(x,\xi)\in\mathbb{R}^{2n}$

$$R_l(x,\xi)=b(x,\xi)-\sum_{|\alpha|<l}a_1^{(\alpha)}(x,\xi)a_{2(\beta)}(x,\xi)/\alpha!, \tag{5}$$

where $a^{(\alpha)}_{(\beta)}(x,\xi)=\partial_\xi^\alpha D_x^\beta a(x,\xi)$. Then we have the following theorem which corresponds to Theorem 18.4.11 in [1].

Theorem 1. *Let $\{a_{j,k}\}_{k=1,2,3,\cdots}$ ($j=1,2$) be sequences in $C_0^\infty(\mathbb{R}^{2n})$ such that the $\{a_{j,k}\}_{k=1,2,3,\cdots}$ are bounded in $S(m_j,g_j)$ and $a_{j,k}\to a_j$ in $C^\infty(\mathbb{R}^{2n})$ as $k\to\infty$. Then $b=a_1\circ a_2$ is well-defined and belongs to $S(m,g)$ and $\{a_{1,k}\circ a_{2,k}\}_{k=1,2,\cdots}$ is a bounded subset of $S(m,g)$ and satisfies*

$$a_{1,k}\circ a_{2,k}\to b\quad \text{in } C^\infty(\mathbb{R}^{2n}) \text{ as } k\to\infty.$$

Moreover, we have $R_l(x,\xi) \in S(mh^l,g)$ *for* $l \in \mathbb{Z}_+$*, where*

$$h(X) = \Big(\sup_{t\in\mathbb{R}^{2n}\setminus\{0\}} \frac{g_{0\,(X,X)}(t)}{g^A_{0\,(X,X)}(t)}\Big)^{1/2}$$

and R_l *is defined by* (5)*.*

Remark. Let $T\colon S(M,G) \to S(m,g)$ be a linear map. We say that T is weakly continuous if for any bounded subset B of $S(M,G)$ TB is bounded in $S(m,g)$ and $T|_B\colon B \to TB$ is continuous with respect to C^∞ topologies of B and TB. From the proof of Theorem 1 the linear map

$$\begin{aligned} T:\ & S(M,G)\cap C_0^\infty(\mathbb{R}^{4n}) \to S(m,g):\\ & \qquad u(x,\xi,y,\eta) \mapsto (\exp(i(A(D_\xi,D_y))u)(x,\xi,x,\xi) \end{aligned}$$

can be extended uniquely to the weakly continuous linear map $\widetilde{T}\colon S(M,G) \to S(m,g)$.

2. Proof of Theorem 1

Let $\varepsilon \in (0,1)$, and put

$$\begin{aligned} \mathscr{P} =& \{\{(X_\nu,Y_\nu)\}_{\nu\in A} \subset \mathbb{R}^{4n};\ A \subset \mathbb{N} \text{ and}\\ & \qquad G_{(X_\nu,Y_\nu)}(X_\nu - X_\mu, Y_\nu - Y_\mu) \ge c_0\varepsilon/C_0 \text{ for } \nu,\mu \in A \text{ with } \nu \ne \mu\}, \end{aligned}$$

where c_0 and C_0 are the constants in (3). $\mathscr{P}$ becomes a partially ordered set by the set inclusion relation and every linearly ordered subset of $\mathscr{P}$ has an upper bound in $\mathscr{P}$. Using Zorn's lemma we can prove the following

Lemma 2 (Lemma 18.4.4 in [1])**.** *For any* $\varepsilon \in (0,1)$ *there are* $\{(X_\nu,Y_\nu)\}_{\nu=1}^\infty \subset \mathbb{R}^{4n}$ *and* $N_\varepsilon \in \mathbb{N}$ *such that*

$$\begin{aligned} &\mathbb{R}^{4n} = \bigcup_{\nu=1}^\infty B_\nu^R \quad \text{if } c_0\varepsilon < R^2,\\ &\bigcap_{j=1}^{N_\varepsilon+1} B_{\nu_j}^R = \emptyset \quad \text{if } R^2 < c_0 \text{ and } 1 \le \nu_1 < \nu_2 < \cdots < \nu_{N_\varepsilon+1}, \end{aligned}$$

where $B_\nu^R = \{(X,Y) \in \mathbb{R}^{4n} : G_{(X_\nu,Y_\nu)}(X - X_\nu, Y - Y_\nu) < R^2\}$ *and* c_0 *is the constant in* (3)*. Moreover, if* $c_0\varepsilon < R^2 < c_0$*, then there are* $\Phi_\nu \in C_0^\infty(B_\nu^R)$ *(* $\nu \in \mathbb{N}$*) and* $C_{k,\varepsilon} > 0$ *(* $k \in \mathbb{Z}_+$*) satisfying* $\sum_{\nu=1}^\infty \Phi_\nu = 1$ *and*

$$|\Phi_\nu|_k^G(X,Y) \le C_{k,\varepsilon} \quad \text{for } (X,Y) \in \mathbb{R}^{4n} \text{ and } \nu \in \mathbb{N}.$$

Note that $M(X,Y)$ is G continuous. Let $u(X,Y) \in S(M,G) \cap C_0^\infty(\mathbb{R}^{4n})$. Put $u_\nu = \Phi_\nu u$. Then we have $u = \sum_{\nu=1}^\infty u_\nu$. Fix $\nu \in \mathbb{N}$, and put

$$K = \{(X,Y) \in \mathbb{R}^{4n};\ G_{(X_\nu,Y_\nu)}(X,Y) < 1\}.$$

By making a linear change of coordinate systems in $\mathbb{R}^{2n}$ (a choice of a basis of $\mathbb{R}^{2n}$) we may assume that $g_{0\,(X_\nu,Y_\nu)}(y,\xi) = |y|^2 + |\xi|^2$. The assumption (A-3) implies that

$$|y|^2 + |\xi|^2 < 1/c \quad \text{if } (x,\eta) \in \mathbb{R}^{2n} \text{ and } (x,\xi,y,\eta) \in K.$$

Therefore, it follows from Sobolev's lemma and Parseval's formula that for each $k \in \mathbb{Z}_+$ there are $C, C' > 0$ such that

$$\begin{aligned}
&\Big|\exp(iA(D_\xi,D_y))v(x,\xi,y,\eta) - \sum_{j<k}(iA(D_\xi,D_y))^j v(x,\xi,y,\eta)/j!\Big| \\
&\le C \sup_{\alpha\in(\mathbb{Z}_+)^{2n},|\alpha|\le n+1} \|((D_y,D_\xi)^\alpha A(D_\xi,D_y)^k v)(x,\cdot,\cdot,\eta)/k!\|_{L^2(\mathbb{R}^{2n})} \\
&\le C' \sup_{j\le n+1} \sup_{(x,\xi^1,y^1,\eta)\in K} |(A(D_\xi,D_y)^k v)(x,\cdot,\cdot,\eta)|_j^{g_{0(X_\nu,Y_\nu)}}(y^1,\xi^1)/k!
\end{aligned}$$

for $v \in C_0^\infty(K)$ and $(x,\xi,y,\eta) \in \mathbb{R}^{4n}$, since

$$|e^w - \sum_{j<k} w^j/j!| \le |w|^k/k! \quad \text{if } \mathrm{Re}\, w \le 0$$

and the volume of $\{(y,\xi) \in \mathbb{R}^{2n};\ (x,\xi,y,\eta) \in K\}$ is less than or equal to $c_{2n}c^{-n}$ for each $(x,\eta) \in \mathbb{R}^{2n}$, where c_{2n} is a positive constant depending only on $2n$. So, under any choice of linear coordinate systems in $\mathbb{R}^{2n}$ we have

$$\begin{aligned}
(6)\qquad &\Big|\exp(iA(D_\xi,D_y))v(x,\xi,y,\eta) - \sum_{j<k}(iA(D_\xi,D_y))^j v(x,\xi,y,\eta)/j!\Big| \\
&\le C_k \sup_{j\le n+1} \sup_{(y^1,\xi^1)} |(A(D_\xi,D_y)^k v)(x,\cdot,\cdot,\eta)|_j^{g_{0(X_\nu,Y_\nu)}}(y^1,\xi^1)
\end{aligned}$$

for $k \in \mathbb{Z}_+$ and $v \in C_0^\infty(K)$. Let $R > 1$, and let $(\hat{x},\hat{\xi},\hat{y},\hat{\eta}) \in \mathbb{R}^{4n}$. First suppose that $(\hat{x},\hat{\xi},\hat{y},\hat{\eta}) \notin R\overline{K}$, where $\overline{K}$ denotes the closure of K in $\mathbb{R}^{4n}$. Define

$$G^A_{(X,Y)}(x,\xi,y,\eta) = \sup_{g_{0(X,Y)}(\tilde{\xi}/2,\tilde{y}/2)<1} \langle(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(x,\xi,y,\eta)\rangle^2$$

for $(x,\xi),(y,\eta),X,Y \in \mathbb{R}^{2n}$. By definition we have

$$G^A_{(X,Y)}(x,\xi,y,\eta) = \infty \quad \text{if } (x,\eta) \ne (0,0),$$

$$G^A_{(X,Y)}(0,\xi,y,0) = g^A_{0\,(X,Y)}(y,\xi).$$

Then there is $a > 0$ satisfying

$$G^A_{(X_\nu,Y_\nu)}(\hat{x}-x,\hat{\xi}-\xi,\hat{y}-y,\hat{\eta}-\eta) \ge a^2 \quad \text{for } (x,\xi,y,\eta) \in RK,$$

since $G^A_{(X_\nu,Y_\nu)}(X,Y) > 0$ if $(X,Y) \ne 0$. We put

$$B = \{(X,Y);\ G^A_{(X_\nu,Y_\nu)}((X,Y)-(\hat{x},\hat{\xi},\hat{y},\hat{\eta})) < a^2\}.$$

It is obvious that

$$\begin{aligned} &G^A_{(X_\nu,Y_\nu)}((X,Y)-(x,\xi,y,\eta))^{1/2} \\ &\ge G^A_{(X_\nu,Y_\nu)}(\hat{x}-x,\hat{\xi}-\xi,\hat{y}-y,\hat{\eta}-\eta)^{1/2} - G^A_{(X_\nu,Y_\nu)}((X,Y)-(\hat{x},\hat{\xi},\hat{y},\hat{\eta}))^{1/2} > 0 \end{aligned}$$

if $(X,Y) \in B$ and $(x,\xi,y,\eta) \in RK$. Therefore, we have $RK \cap B = \emptyset$, and there is $(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in \mathbb{R}^{4n}$ such that

$$\langle(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(x,\xi,y,\eta)\rangle < \langle(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(\hat{x},\hat{\xi},\hat{y},\hat{\eta})+(X,Y)\rangle \tag{7}$$

if $(x,\xi,y,\eta) \in RK$, $(X,Y) \in \mathbb{R}^{4n}$ and $G^A_{(X_\nu,Y_\nu)}(X,Y) < a^2$. From the bipolar theorem we have

$$\langle(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(x,\xi,y,\eta)\rangle \le \langle(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\rangle - a g_{0\,(X_\nu,Y_\nu)}(\tilde{\xi}/2,\tilde{y}/2)^{1/2} \tag{8}$$

if $(x,\xi,y,\eta) \in RK$. Indeed, putting

$$\begin{aligned} B_1 &= \{(x,\xi,y,\eta) \in \mathbb{R}^{4n};\ G^A_{(X_\nu,Y_\nu)}(x,\xi,y,\eta) \le 1\}, \\ B_2 &= \{(x,\xi,y,\eta) \in \mathbb{R}^{4n};\ g_{0\,(X_\nu,Y_\nu)}(\xi/2,y/2) \le 1\}, \end{aligned}$$

we have

$$B_1 = \{(0,\xi,y,0) \in \mathbb{R}^{4n};\ g^A_{0\,(X_\nu,Y_\nu)}(y,\xi) \le 1\}$$

and B_2 is a closed convex set and satisfies

$$B_2 = -B_2 \text{ and } 0 \in \mathring{B}_2, \tag{9}$$

where $\mathring{B}_2$ denotes the interior of B_2. Define

$$\begin{aligned} B_j^* = &\{(x,\xi,y,\eta) \in \mathbb{R}^{4n};\ |\langle(x,\xi,y,\eta),(x^1,\xi^1,y^1,\eta^1)\rangle| \le 1 \\ &\text{for any } (x^1,\xi^1,y^1,\eta^1) \in B_j\} \quad (j = 1,2). \end{aligned}$$

Then we have

$$B_2^* = \{(0,\xi,y,0) \in \mathbb{R}^{4n};\ \sup_{(y^1,\xi^1)} \langle (y,\xi),(y^1,\xi^1)\rangle^2 / g_{0\,(X_\nu,Y_\nu)}(\xi^1/2, y^1/2) \le 1\}$$
$$= \{(0,\xi,y,0) \in \mathbb{R}^{4n};\ g^A_{0\,(X_\nu,Y_\nu)}(y,\xi) \le 1\} = B_1.$$

Similarly, we have

$$B_1^* = \{(x,\xi,y,\eta) \in \mathbb{R}^{4n};\ \sup_{(y^1,\xi^1)} \langle (y,\xi),(y^1,\xi^1)\rangle^2 / g^A_{0\,(X_\nu,Y_\nu)}(y^1,\xi^1) \le 1\}.$$

It is obvious that $B_2 \subset B_1^*$. Now suppose that $B_1^* \setminus B_2 \neq \emptyset$. Then, by the Hahn-Banach theorem (the Mazur theorem) and (9) there are $(y^0,\xi^0),(y^1,\xi^1) \in \mathbb{R}^{2n}$ such that $(0,\xi^0,y^0,0) \in B_1^* \setminus B_2\,(= (B_2^*)^* \setminus B_2)$ and

$$\langle (y^0,\xi^0),(y^1,\xi^1)\rangle > \sup_{(0,\xi,y,0)\in B_2} \langle (y,\xi),(y^1,\xi^1)\rangle > 0.$$

Putting $(y^2,\xi^2) = \Big(\sup_{(0,\xi,y,0)\in B_2} \langle (y,\xi),(y^1,\xi^1)\rangle\Big)^{-1} (y^1,\xi^1)$, we have $(0,\xi^2,y^2,$ $0) \in B_2^*\,(= B_1)$ and

$$1 \ge \langle (y^0,\xi^0),(y^2,\xi^2)\rangle > 1,$$

which leads to contradiction. So we have $B_1^*\,(= (B_2^*)^*) = B_2$ (the bipolar theorem), *i.e.*,

$$g_{0\,(X_\nu,Y_\nu)}(\xi/2,y/2) = \sup_{(y^1,\xi^1)} \langle (y,\xi),(y^1,\xi^1)\rangle^2 / g^A_{0\,(X_\nu,Y_\nu)}(y^1,\xi^1). \tag{10}$$

Therefore, we have

$$\inf_{G^A_{(X_\nu,Y_\nu)}(X,Y)<a^2} \langle (\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(X,Y)\rangle$$
$$= -a \sup_{(X,Y)} \langle (\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(X,Y)\rangle / G^A_{(X_\nu,Y_\nu)}(X,Y)^{1/2}$$
$$= -a\Big\{\sup_{(X,Y)} \langle (\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}),(X,Y)\rangle^2 / G^A_{(X_\nu,Y_\nu)}(X,Y)\Big\}^{1/2}$$
$$= -a\Big\{\sup_{(y^1,\xi^1)} \langle (\tilde{y},\tilde{\xi}),(y^1,\xi^1)\rangle^2 / g^A_{0\,(X_\nu,Y_\nu)}(y^1,\xi^1)\Big\}^{1/2}$$
$$= -a g_{0\,(X_\nu,Y_\nu)}(\tilde{\xi}/2,\tilde{y}/2)^{1/2}.$$

This, together with (7), gives (8). Put

$$L(x,\xi,y,\eta) = \langle \tilde{x}, x-\hat{x}\rangle + \langle \tilde{\xi}, \xi-\hat{\xi}\rangle + \langle \tilde{y}, y-\hat{y}\rangle + \langle \tilde{\eta}, \eta-\hat{\eta}\rangle. \tag{11}$$

Then we can see that

$$(12)\qquad |L(0)/L(x,\cdot,\cdot,\eta)|_k^{g_{0(X_V,Y_V)}}(y,\xi) \le k!R/(R-1)^{k+1} \quad \text{for } (x,\xi,y,\eta)\in K \text{ and } k\in\mathbb{Z}_+.$$

Indeed, it follows from (7) with $(X,Y)=0$ and Lemma 18.4.5 in [1] that $L(x,\xi,y,\eta)\neq 0$ and

$$(13)\qquad |L(0)/L|_k^{G_{(X_V,Y_V)}}(x,\xi,y,\eta) \le k!R/(R-1)^{k+1} \quad \text{for } (x,\xi,y,\eta)\in RK.$$

On the other hand, we have

$$|f(x,\cdot,\cdot,\eta)|_k^{g_{0(X_V,Y_V)}}(y,\xi) \le |f|_k^{G_{(X_V,Y_V)}}(x,\xi,y,\eta),$$

which proves (12). Since

$$[\exp(iA(X_\xi,X_y)),\langle\tilde{\xi},i\partial_{X_\xi}\rangle] = \langle\tilde{\xi},X_y\rangle\exp(iA(X_\xi,X_y)) \quad \text{for } (X_\xi,X_y)\in\mathbb{R}^{2n},$$

we have

$$[\exp(iA(D_\xi,D_y)),L(x,\xi,y,\eta)]v = \exp(iA(D_\xi,D_y))(\langle\tilde{\xi},D_y\rangle+\langle\tilde{y},D_\xi\rangle)v,$$

where $[T,S]v=T(Sv)-S(Tv)$ and $\langle\tilde{\xi},D_y\rangle=\sum_{j=1}^n\tilde{\xi}_jD_{y_j}$. So we have

$$\begin{aligned}&\exp(iA(D_\xi,D_y))(L(x,\xi,y,\eta)v(x,\xi,y,\eta))|_{(x,\xi,y,\eta)=(\hat{x},\hat{\xi},\hat{y},\hat{\eta})}\\ &=\exp(iA(D_\xi,D_y))(\langle\tilde{\xi},D_y\rangle+\langle\tilde{y},D_\xi\rangle)v(x,\xi,y,\eta)|_{(x,\xi,y,\eta)=(\hat{x},\hat{\xi},\hat{y},\hat{\eta})},\end{aligned}$$

since $L(\hat{x},\hat{\xi},\hat{y},\hat{\eta})=0$. Replacing v by $L^{-1}v$, we have

$$\begin{aligned}&(\exp(iA(D_\xi,D_y))v)(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\\ &=\exp(iA(D_\xi,D_y))(\langle\tilde{\xi},D_y\rangle+\langle\tilde{y},D_\xi\rangle)L(x,\xi,y,\eta)^{-1}v|_{(x,\xi,y,\eta)=(\hat{x},\hat{\xi},\hat{y},\hat{\eta})}.\end{aligned}$$

Therefore, by induction we have

$$\begin{aligned}(14)\quad &(\exp(iA(D_\xi,D_y))v)(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\\ &=\exp(iA(D_\xi,D_y))(((\langle\tilde{\xi},D_y\rangle+\langle\tilde{y},D_\xi\rangle)L(x,\xi,y,\eta)^{-1})^kv|_{(x,\xi,y,\eta)=(\hat{x},\hat{\xi},\hat{y},\hat{\eta})}\end{aligned}$$

for $k\in\mathbb{N}$. By (13) and induction on k we see that for any $j\in\mathbb{Z}_+$ and $k\in\mathbb{N}$ there is $C_{j,k,R}>0$ such that

$$(15)\quad |((((\langle\tilde{\xi},D_y\rangle+\langle\tilde{y},D_\xi\rangle)L(\hat{x},\cdot,\cdot,\hat{\eta})^{-1})^kv)(\hat{x},\cdot,\cdot,\hat{\eta})|_j^{g_{0(X_V,Y_V)}}(y^1,\xi^1)$$

$$\leq C_{j,k,R}\Big(g_{0\,(X_\nu,Y_\nu)}(\tilde{\xi}/2,\tilde{y}/2)^{1/2}/|L(0)|\Big)^k \sup_{l\leq k+j}|v(\hat{x},\cdot,\cdot,\hat{\eta})|_l^{g_{0\,(X_\nu,Y_\nu)}}(y^1,\xi^1)$$

$$\text{for } (y^1,\xi^1)\in\mathbb{R}^{2n}.$$

(6) with $k=0$, (14) and (15) yield, with $C_{k,R}>0$,

$$|(\exp(iA(D_\xi,D_y))v)(\hat{x},\hat{\xi},\hat{y},\hat{\eta})| \leq C_{k,R}(g_{0\,(X_\nu,Y_\nu)}(\tilde{\xi}/2,\tilde{y}/2)^{1/2}/|L(0)|)^k \times \sup_{j\leq n+1+k}\sup_{(y^1,\xi^1)}|v(x,\cdot,\cdot,\eta)|_j^{g_{0\,(X_\nu,Y_\nu)}}(y^1,\xi^1) \tag{16}$$

for $v\in C_0^\infty(K)$. It follows from (8) with $(x,\xi,y,\eta)=0$ that

$$|L(0)|\geq a g_{0\,(X_\nu,Y_\nu)}(\tilde{\xi}/2,\tilde{y}/2)^{1/2},$$

since $g_{0\,(X_\nu,Y_\nu)}\geq 0$. Therefore, exchanging $(\hat{x},\hat{\xi},\hat{y},\hat{\eta})$ with (x,ξ,y,η) and taking $a=\inf_{(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\in RK} G^A_{(X_\nu,Y_\nu)}(x-\hat{x},\xi-\hat{\xi},y-\hat{y},\eta-\hat{\eta})^{1/2}$, from (16) we have

$$\begin{aligned} &|\exp(iA(D_\xi,D_y))v(x,\xi,y,\eta)| \\ &\leq C_{k,R}(1+\inf_{(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\in RK} G^A_{(X_\nu,Y_\nu)}(x-\hat{x},\xi-\hat{\xi},y-\hat{y},\eta-\hat{\eta}))^{-k/2} \\ &\quad\times \sup_{j\leq n+1+k}\sup_{(y^1,\xi^1)}|v(x,\cdot,\cdot,\eta)|_j^{g_{0\,(X_\nu,Y_\nu)}}(y^1,\xi^1) \end{aligned} \tag{17}$$

for $k\in\mathbb{Z}_+$, $R>1$, $v\in C_0^\infty(K)$ and $(x,\xi,y,\eta)\notin R\overline{K}$. We note that $\exp(iA(D_\xi,D_y))\times v(x,\xi,y,\eta)=0$ for $v\in C_0^\infty(K)$ if $(x,\xi^1,y^1,\eta)\notin K$ for any $(y^1,\xi^1)\in\mathbb{R}^{2n}$ and that $\inf_{(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\in RK} G^A_{(X_\nu,Y_\nu)}(x-\hat{x},\xi-\hat{\xi},y-\hat{y},\eta-\hat{\eta})=\inf_{(x,\hat{\xi},\hat{y},\eta)\in RK} g^A_{0\,(X_\nu,Y_\nu)}(y-\hat{y},\xi-\hat{\xi})<\infty$ if $(x,\xi^1,y^1,\eta)\in RK$ for some $(y^1,\xi^1)\in\mathbb{R}^{2n}$. If $(x,\xi,y,\eta)\in R\overline{K}$, then by (6) with $k=0$ (17) is also valid, since $\inf_{(\hat{x},\hat{\xi},\hat{y},\hat{\eta})\in RK} G^A_{(X_\nu,Y_\nu)}(x-\hat{x},\xi-\hat{\xi},y-\hat{y},\eta-\hat{\eta})=0$ for $(x,\xi,y,\eta)\in R\overline{K}$. Therefore, we have

$$\begin{aligned} &|(\exp(iA(D_\xi,D_y))v)(x,\xi,x,\xi)| \\ &\leq C_{k,R}(1+\inf_{(x,\eta,y,\xi)\in RK} g^A_{0\,(X_\nu,Y_\nu)}(x-y,\xi-\eta))^{-k/2} \\ &\quad\times \sup_{j\leq n+1+k}\sup_{(y,\eta)}|v(x,\cdot,\cdot,\xi)|_j^{g_{0\,(X_\nu,Y_\nu)}}(y,\eta) \end{aligned} \tag{18}$$

for $k\in\mathbb{Z}_+$, $R>1$, $v\in C_0^\infty(K)$ and $(x,\xi)\in\mathbb{R}^{2n}$.

Lemma 3. *For $X,Y,(y,\xi)\in\mathbb{R}^{2n}$ we have*

$$\begin{aligned} &g^A_{0\,(X,Y)}(y,\xi)\,(=G^A_{(X,Y)}(0,\xi,y,0)=4g^\sigma_{0\,(X,Y)}(y,\xi)) \\ &=4\sup_{\tilde{\xi}\neq 0}\frac{\langle\tilde{\xi},y\rangle^2}{g_{1X}(0,\tilde{\xi})}+4\sup_{\tilde{y}\neq 0}\frac{\langle\tilde{y},\xi\rangle^2}{g_{2Y}(\tilde{y},0)}\leq 4g^\sigma_{1X}(y,0)+4g^\sigma_{2Y}(0,\xi). \end{aligned}$$

Proof. By definition we have

$$G^A_{(X,Y)}(0,\xi,y,0) = 4\sup_{(\tilde{y},\tilde{\xi})}\frac{(\langle\tilde{\xi},y\rangle+\langle\tilde{y},\xi\rangle)^2}{g_{1X}(0,\tilde{\xi})+g_{2Y}(\tilde{y},0)} \tag{19}$$
$$\leq 4\sup_{\tilde{\xi}\neq 0}\frac{\langle\tilde{\xi},y\rangle^2}{g_{1X}(0,\tilde{\xi})}+4\sup_{\tilde{y}\neq 0}\frac{\langle\tilde{y},\xi\rangle^2}{g_{2Y}(\tilde{y},0)}.$$
$$\leq 4g^\sigma_{1X}(y,0)+4g^\sigma_{2Y}(0,\xi),$$

since

$$\frac{(a+b)^2}{c+d}\leq\frac{a^2}{c}+\frac{b^2}{d}\quad \text{if } a,b,c,d\geq 0 \text{ and } c+d>0,$$

where $\alpha/0=\infty$ for $\alpha\geq 0$. Put

$$\mathscr{A}=\sup_{\tilde{\xi}\neq 0}\frac{\langle\tilde{\xi},y\rangle^2}{g_{1X}(0,\tilde{\xi})},\quad \mathscr{B}=\sup_{\tilde{y}\neq 0}\frac{\langle\tilde{y},\xi\rangle^2}{g_{2Y}(\tilde{y},0)}$$

for a fixed $(y,\xi)\in\mathbb{R}^{2n}$. Then there is $(\hat{y},\hat{\xi})\in\mathbb{R}^{2n}$ such that

$$\mathscr{A}=\frac{\langle\hat{\xi},y\rangle^2}{g_{1X}(0,\hat{\xi})},\quad \mathscr{B}=\frac{\langle\hat{y},\xi\rangle^2}{g_{2Y}(\hat{y},0)}.$$

From (19) we have

$$G^A_{(X,Y)}(0,\xi,y,0)\geq 4\frac{(\langle\mu\hat{\xi},y\rangle+\langle\lambda\hat{y},\xi\rangle)^2}{g_{1X}(0,\mu\hat{\xi})+g_{2Y}(\lambda\hat{y},0)}$$
$$=4\frac{(\mu\mathscr{A}^{1/2}g_{1X}(0,\hat{\xi})^{1/2}+\lambda\mathscr{B}^{1/2}g_{2Y}(\hat{y},0)^{1/2})^2}{\mu^2 g_{1X}(0,\hat{\xi})+\lambda^2 g_{2Y}(\hat{y},0)}$$

for $\lambda,\mu>0$. Taking $\mu=\mathscr{A}^{1/2}/g_{1X}(0,\hat{\xi})^{1/2}$ and $\lambda=\mathscr{B}^{1/2}/g_{2Y}(\hat{y},0)^{1/2}$ we have

$$G^A_{(X,Y)}(0,\xi,y,0)\geq 4(\mathscr{A}+\mathscr{B}),$$

which gives $G^A_{(X,Y)}(0,\xi,y,0)=4(\mathscr{A}+\mathscr{B})$. □

We note that Lemma 3 and (A-3) yield

$$4c(g^\sigma_{1X}(y,0)+g^\sigma_{2Y}(0,\xi))\leq g^A_{0(X,Y)}(y,\xi)\leq 4(g^\sigma_{1X}(y,0)+g^\sigma_{2Y}(0,\xi)) \tag{20}$$

for $X,Y,(y,\xi)\in\mathbb{R}^{2n}$.

Lemma 4 (Proposition 18.5.3 in [1]). *g is σ temperate and m_j ($j = 1,2$) are σ, g temperate (under the assumptions* (A-1)–(A-3))*. Moreover, G is uniformly A temperate in $\Delta \equiv \{(x,\xi,x,\xi);\ (x,\xi) \in \mathbb{R}^{2n}\}$, i.e., G is slowly varying, and there are $C(G) > 0$ and $N(G)$ such that*

$$G_{(x,\eta,y,\xi)}(X,Y) \le C(G) G_{(x,\xi,x,\xi)}(X,Y)(1 + g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{N(G)} \tag{21}$$

for $(x,\xi),(y,\eta),X,Y \in \mathbb{R}^{2n}$.

Proof. Let F_1 and F_2 be positive definite quadratic forms on the vector space V ($\equiv \mathbb{R}^{2n}$). Define the dual forms F'_j ($j = 1,2$) on the dual space V' ($\cong \mathbb{R}^{2n}$) by

$$F'_j(\widetilde{X}) = \sup_{X \in V \setminus \{0\}} \frac{\langle \widetilde{X}, X\rangle^2}{F_j(X)} \quad \text{for } \widetilde{X} \in V'.$$

Then we can see that

$$(F_1 + F_2)'(\widetilde{X}) = \inf_{\tilde{t} \in V'} (F'_1(\widetilde{X} - \tilde{t}) + F'_2(\tilde{t})) \quad (\ \widetilde{X} \in V'). \tag{22}$$

Indeed, we can choose a basis of V so that $F_1(X)$ is represented as $F_1(X) = \sum_{j=1}^{2n} X_j^2$. Moreover, we can choose an orthonormal basis of V so that

$$F_1(X) = \sum_{j=1}^{2n} X_j^2, \quad F_2(X) = \sum_{j=1}^{2n} a_j X_j^2 \quad (\ X \in V),$$

where $a_j > 0$. Then by the dual basis of V' $F'_1(\widetilde{X})$ and $F'_2(\widetilde{X})$ are represented as

$$F'_1(\widetilde{X}) = \sum_{j=1}^{2n} \widetilde{X}_j^2, \quad F'_2(\widetilde{X}) = \sum_{j=1}^{2n} a_j^{-1} \widetilde{X}_j^2,$$
$$(F_1 + F_2)'(\widetilde{X}) = \sum_{j=1}^{2n} (1 + a_j)^{-1} \widetilde{X}_j^2 \quad (\ \widetilde{X} \in V').$$

On the other hand, we have

$$\begin{aligned} &\inf_{\tilde{t} \in V'} (F'_1(\widetilde{X} - \tilde{t}) + F'_2(\tilde{t})) \\ &= \sum_{j=1}^{2n} \inf_{\tilde{t}_j \in \mathbb{R}} ((\widetilde{X}_j - \tilde{t}_j)^2 + a_j^{-1} \tilde{t}_j^2) = \sum_{j=1}^{2n} (1 + a_j)^{-1} \widetilde{X}_j^2 \quad (\ \widetilde{X} \in V'). \end{aligned}$$

This proves (22). Therefore, we have

$$g^\sigma_X(Y) = \inf_{t \in \mathbb{R}^{2n}} 2(g^\sigma_{1X}(Y - t) + g^\sigma_{2X}(t)) \quad \text{for } X,Y \in \mathbb{R}^{2n}. \tag{23}$$

First let us prove that there are $C>0$ and N satisfying

$$g_{jX}(t) \leq C g_{jY}(t)(1+g_X^\sigma(X-Y))^N \quad (j=1,2), \tag{24}$$

which implies that g is σ temperate. From (23) it follows that (24) is valid if and only if, with some $C'>0$,

$$g_{jX}(t) \leq C' g_{jY}(t) M^N, \quad M = 1+g_{1X}^\sigma(Y-t_0)+g_{2X}^\sigma(t_0-X) \tag{24$'$}$$

for any $X,Y,t,t_0 \in \mathbb{R}^{2n}$ and $j=1,2$. Applying the bipolar theorem, we have

$$(g_{jX}^\sigma)^\sigma = g_{jX} \quad (j=1,2,\ X\in\mathbb{R}^{2n}) \quad \text{(see the proof of (10))}.$$

So (A-1) implies that

$$\begin{aligned} g_{1X}(t) &\leq C(g_1,g_2) g_{1Y}(t)(1+g_{2Y}^\sigma(X-Y))^{N(g_1,g_2)}, \\ g_{2X}(t) &\leq C(g_1,g_2) g_{2Y}(t)(1+g_{1Y}^\sigma(X-Y))^{N(g_1,g_2)} \end{aligned}$$

for $X,Y,t \in \mathbb{R}^{2n}$. This gives

$$\begin{aligned} & g_{1X}(t) \leq C(g_1,g_2) g_{1t_0}(t)(1+g_{2t_0}^\sigma(t_0-X))^{N(g_1,g_2)} \\ & \quad \leq C'(g_1,g_2) g_{1t_0}(t) M^{(N(g_2)+1)N(g_1,g_2)}, \\ & g_{2X}(t) \leq C_1(g_2) g_{2t_0}(t)(1+g_{2X}^\sigma(t_0-X))^{N(g_2)} \leq C_1(g_2) g_{2t_0}(t) M^{N(g_2)}, \\ & g_{1t_0}(t) \leq C_1(g_1) g_{1Y}(t)(1+g_{1t_0}^\sigma(Y-t_0))^{N(g_1)}, \\ & g_{2t_0}(t) \leq C(g_1,g_2) g_{2Y}(t)(1+g_{1Y}^\sigma(Y-t_0))^{N(g_1,g_2)} \\ & \quad \leq C'(g_1,g_2) g_{2Y}(t)(1+g_{1t_0}^\sigma(Y-t_0))^{(N(g_1)+1)N(g_1,g_2)}, \end{aligned}$$

$$\begin{aligned} & 1+g_{1t_0}^\sigma(Y-t_0) \leq C(g_1,g_2)(1+g_{1X}^\sigma(Y-t_0))(1+g_{2X}^\sigma(t_0-X))^{N(g_1,g_2)} \\ & \quad \leq C(g_1,g_2) M^{N(g_1,g_2)+1} \end{aligned} \tag{25}$$

for $X,Y,t_0,t \in \mathbb{R}^{2n}$, since the g_j are σ temperate and

$$1+g_{2t_0}^\sigma(t_0-X) \leq C_1(g_2)(1+g_{2X}^\sigma(t_0-X))^{N(g_2)+1} \leq C_1(g_2) M^{N(g_2)+1}, \tag{26}$$

$$1+g_{1Y}^\sigma(Y-t_0) \leq C_1(g_1)(1+g_{1t_0}^\sigma(Y-t_0))^{N(g_1)+1}. \tag{27}$$

Therefore, we have

$$\begin{aligned} g_{1X}(t) \leq & C'(g_1,g_2) C_1(g_1) C(g_1,g_2)^{N(g_1)} g_{1Y}(t) \\ & \times M^{(N(g_1)+N(g_2)+1)N(g_1,g_2)+N(g_1)}, \end{aligned}$$

$$g_{2X}(t) \leq C'(g_1,g_2)C_1(g_2)C(g_1,g_2)^{(N(g_1)+1)N(g_1,g_2)}g_{2Y}(t) \\ \times M^{(N(g_1)+1)(N(g_1,g_2)+1)N(g_1,g_2)+N(g_2)}$$

for $X,Y,t \in \mathbb{R}^{2n}$, which proves (24)$'$ and (24). It is obvious that the m_j are g continuous. Let us repeat the same argument as for g in order to prove that the m_j are σ, g temperate. For this purpose it suffices to show that there are $C_1 > 0$ and N_1 such that

$$m_j(X) \leq C_1 m_j(Y) M^{N_1}, \quad M = 1 + g^{\sigma}_{1X}(Y - t_0) + g^{\sigma}_{2X}(t_0 - X)$$

for any $X,Y,t_0 \in \mathbb{R}^{2n}$ and $j = 1,2$. From (A-2) and (25) – (27) we have

$$\begin{aligned} m_1(X) \leq& C m_1(t_0)(1 + g^{\sigma}_{2t_0}(t_0 - X))^N \leq CC_1(g_2)^N m_1(t_0) M^{(N(g_2)+1)N}, \\ m_1(t_0) \leq& C_1(m_1) m_1(Y)(1 + g^{\sigma}_{1t_0}(Y - t_0))^{N(m_1)} \\ \leq& C_1(m_1)C(g_1,g_2)^{N(m_1)} m_1(Y) M^{N(m_1)(N(g_1,g_2)+1)} \end{aligned}$$

for $X,Y,t_0 \in \mathbb{R}^{2n}$ and, therefore,

$$m_1(X) \leq C_1 m_1(Y) M^{N_1}.$$

Similarly, we have

$$\begin{aligned} & m_2(X) \leq C_1(m_2) m_2(t_0)(1 + g^{\sigma}_{2X}(t_0 - X))^{N(m_2)} \leq C_1(m_2) m_2(t_0) M^{N(m_2)}, \\ & m_2(t_0) \leq C m_2(Y)(1 + g^{\sigma}_{1Y}(Y - t_0))^N \\ & \leq CC_1(g_1)^N m_2(Y)(1 + g^{\sigma}_{1t_0}(Y - t_0))^{N(N(g_1)+1)} \\ & \leq CC_1(g_1)^N C(g_1,g_2)^{N(N(g_1)+1)} m_2(Y) M^{N(N(g_1)+1)(N(g_1,g_2)+1)}, \\ & m_2(X) \leq C_1 m_2(Y) M^{N_1} \end{aligned}$$

for $X,Y,t_0 \in \mathbb{R}^{2n}$, which proves that the m_j are σ, g temperate. Moreover, (24) yields

(28)
$$g_{1(x,\eta)}(X) + g_{2(y,\xi)}(Y) \\ \leq C(g_{1(x,\xi)}(X) + g_{2(x,\xi)}(Y))(1 + g^{\sigma}_{(x,\eta)}(0, \xi - \eta) + g^{\sigma}_{(y,\xi)}(x - y, 0))^N$$

for $(x,\eta),(y,\xi),X,Y \in \mathbb{R}^{2n}$. Put

$$\widetilde{M} = 1 + g^{\sigma}_{(x,\eta)}(x - y, 0) + g^{\sigma}_{(y,\xi)}(0, \xi - \eta).$$

Then we have

$$g^{\sigma}_{(y,\eta)}(x - y, 0) \leq C g^{\sigma}_{(x,\eta)}(x - y, 0)(1 + g^{\sigma}_{(x,\eta)}(x - y, 0))^N \leq C\widetilde{M}^{N+1},$$

$$
\begin{aligned}
g^\sigma_{(y,\eta)}(0,\xi-\eta) &\le C g^\sigma_{(y,\xi)}(0,\xi-\eta)(1+g^\sigma_{(y,\xi)}(0,\xi-\eta))^N \le C\widetilde{M}^{N+1},\\
g^\sigma_{(x,\eta)}(0,\xi-\eta) &\le C g^\sigma_{(y,\eta)}(0,\xi-\eta)(1+g^\sigma_{(y,\eta)}(x-y,0))^N \le C'\widetilde{M}^{(N+1)^2},\\
g^\sigma_{(y,\xi)}(x-y,0) &\le C g^\sigma_{(y,\eta)}(x-y,0)(1+g^\sigma_{(y,\eta)}(0,\xi-\eta))^N \le C'\widetilde{M}^{(N+1)^2},
\end{aligned}
$$

since g is σ temperate. This, together with (20) and (28), gives, with some $C>0$,

$$
G_{(x,\eta,y,\xi)}(X,Y) \le C G_{(x,\xi,x,\xi)}(X,Y)(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{N(N+1)^2},
$$

since $g^\sigma_X \le 2g^\sigma_{jX}$. Therefore, G is uniformly A temperate in Δ. □

Let $R, R_0 \in \mathbb{R}$ satisfy $0<R<R_0<c_0^{1/2}$, and put

$$
\begin{aligned}
U_\nu &= \{(X,Y)\in\mathbb{R}^{4n};\ G_{(X_\nu,Y_\nu)}(X-X_\nu,Y-Y_\nu)\le R_0^2\},\\
U'_\nu &= \{(X,Y)\in\mathbb{R}^{4n};\ G_{(X_\nu,Y_\nu)}(X-X_\nu,Y-Y_\nu)\le c_0\}.
\end{aligned}
$$

Let us apply (18) to $v=\Phi_\nu u\,(\equiv u_\nu)$ with K, R and $G_{(X_\nu,Y_\nu)}$ replaced by B^R_ν, R_0/R and $G_{(X_\nu,Y_\nu)}/R^2$, respectively.

Lemma 5 (Lemma 18.4.8 in [1])**.** *There are $C_1>0$ and N_1 satisfying*

$$
\sum_{\nu=1}^{\infty}(1+d_\nu(x,\xi))^{-N_1}\le C_1 \quad \textit{for } (x,\xi)\in\mathbb{R}^{2n},
$$

where

$$
d_\nu(x,\xi)=\begin{cases}\inf_{(x,\eta,y,\xi)\in U_\nu} g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta)\\ \qquad\qquad \textit{if there is } (y^1,\eta^1)\in\mathbb{R}^{2n} \textit{ satisfying } (x,\eta^1,y^1,\xi)\in U_\nu,\\ \infty \quad \textit{otherwise}.\end{cases}
$$

Proof. Let us repeat the proof in [1] again. Let $(x,\xi)\in\mathbb{R}^{2n}$. We may assume that $g_{0\,(x,\xi,x,\xi)}$ is the square of the Euclidean norm $|\cdot|$ of $\mathbb{R}^{2n}$. Then (A-4) implies that

$$
|t|^2\le g^A_{0\,(x,\xi x,\xi)}(t) \quad (\,t\in\mathbb{R}^{2n}).
$$

Let $k\in\mathbb{N}$, and put

$$
M_k=\{\nu\in\mathbb{N};\ d_\nu(x,\xi)\le k\}.
$$

By definition, for every $\nu\in M_k$ there is $(\hat{y}_\nu,\hat{\eta}_\nu)\in\mathbb{R}^{2n}$ satisfying $(x,\hat{\eta}_\nu,\hat{y}_\nu,\xi)\in U_\nu$ and

$$
g^A_{0\,(x,\hat{\eta}_\nu,\hat{y}_\nu,\xi)}(x-\hat{y}_\nu,\xi-\hat{\eta}_\nu)\le k.
$$

Since G is uniformly A temperate in Δ, we have

$$
C_0^{-1}\le g_{0\,(X_\nu,Y_\nu)}/g_{0\,(x,\hat{\eta}_\nu,\hat{y}_\nu,\xi)}\le C_0,
$$

(29) $g_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(t) \le C(G) g_{0(x,\xi,x,\xi)}(t)(1+g^A_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(x-\hat y_\nu,\xi-\hat\eta_\nu))^{N(G)}$
$\le (2k)^{N(G)}C(G)|t|^2 \quad \text{for } t\in\mathbb{R}^{2n}.$

Put, with $c_1>0$,

$$V_\nu = \{(y,\eta)\in\mathbb{R}^{2n};\ |(y-\hat y_\nu,\eta-\hat\eta_\nu)|<c_1k^{-N(G)/2}\} \quad \text{for } \nu\in M_k.$$

If $(y,\eta)\in V_\nu$, then by (29) we have

$$g_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(y-\hat y_\nu,\eta-\hat\eta_\nu)\le 2^{N(G)}C(G)c_1^2.$$

Choose $c_1>0$ so that $2^{N(G)}C_0^2C(G)c_1^2<(c_0^{1/2}-R_0)^2$. Then we have

$$\begin{aligned}&G_{(X_\nu,Y_\nu)}(x-x_\nu,\eta-\xi_\nu,y-y_\nu,\xi-\eta_\nu)^{1/2}\\ &\le G_{(X_\nu,Y_\nu)}(x-x_\nu,\hat\eta_\nu-\xi_\nu,\hat y_\nu-y_\nu,\xi-\eta_\nu)^{1/2}+g_{0(X_\nu,Y_\nu)}(y-\hat y_\nu,\eta-\hat\eta_\nu)^{1/2}\\ &\le R_0+C_0g_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(y-\hat y_\nu,\eta-\hat\eta_\nu)^{1/2}<c_0^{1/2}\quad \text{if } (y,\eta)\in V_\nu,\end{aligned}$$

where $X_\nu=(x_\nu,\xi_\nu)$ and $Y_\nu=(y_\nu,\eta_\nu)$. This implies that

(30) $$V_\nu\subset\{(y,\eta)\in\mathbb{R}^{2n};\ (x,\eta,y,\xi)\in U'_\nu\}.$$

Since

(31) $$g^A_{0(x,\xi,x,\xi)}(t)\le C(G)g^A_{0(x,\eta,y,\xi)}(t)(1+g^A_{0(x,\eta,y,\xi)}(x-y,\xi-\eta))^{N(G)},$$

we have

$$\begin{aligned}&|(x-\hat y_\nu,\xi-\hat\eta_\nu)|^2\le g^A_{0(x,\xi,x,\xi)}(x-\hat y_\nu,\xi-\hat\eta_\nu)\\ &\le C(G)(1+g^A_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(x-\hat y_\nu,\xi-\hat\eta_\nu))^{N(G)+1}\le(2k)^{N(G)+1}C(G).\end{aligned}$$

So there is $C>0$ such that

(32) $$V_\nu\subset\{(y,\eta)\in\mathbb{R}^{2n};\ |(y-x,\eta-\xi)|<Ck^{(N(G)+1)/2}\}.$$

With $\varepsilon=1/2$ in Lemma 2 the number of U'_ν which can overlap is not greater than N_ε. Therefore, by (30) and (32) there are positive constants C, C' and c such that

$$c|M_k|k^{-nN(G)}\le\sum_{\nu\in M_k}\mu(V_\nu)\le C\mu\Big(\bigcup_{\nu\in M_k}V_\nu\Big)\le C'k^{n(N(G)+1)},$$

where $|M_k|$ denotes the number of the elements in M_k and μ denotes the Lebesgue measure in $\mathbb{R}^{2n}$. Putting $N_1=[2nN(G)+n]+2$, we have, with some $C,C_1>0$,

$$|M_k|\le Ck^{N_1-1},$$

$$\sum_{\nu\in\mathbb{N}} (1+d_\nu(x,\xi))^{-N_1} \leq \sum_{\nu\in M_1} 1^{-N_1} + \sum_{k=1}^{\infty} \sum_{\nu\in M_{2^k}\setminus M_{2^{k-1}}} (1+2^{k-1})^{-N_1}$$
$$\leq C\Big(1+\sum_{k=1}^{\infty} 2^{k(N_1-1)}(1+2^{k-1})^{-N_1}\Big) \leq C\Big(1+\sum_{k=1}^{\infty} 2^{N_1-k}\Big) \leq C_1,$$

where $[\kappa]$ denotes the largest integer $\leq \kappa$. □

Since

$$d_\nu(x,\xi) \leq C_0 g^A_{0\,(X_\nu,Y_\nu)}(x-y,\xi-\eta) \quad \text{if } (x,\eta,y,\xi)\in U_\nu,$$
$$C_0^{-1} \leq g_{0\,(x,\eta,y,\xi)}/g_{0\,(X_\nu,Y_\nu)} \leq C_0 \quad \text{if } (x,\eta,y,\xi)\in \operatorname{supp} u_\nu,$$

(18) yields, with some $C'_{k,R} > 0$,

(33) $$|(\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,x,\xi)| \leq C'_{k,R}(1+d_\nu(x,\xi))^{-k/2} \times \sup_{j\leq 2n+1+k}\sup_{(y,\eta)} |u_\nu(x,\cdot,\cdot,\xi)|_j^{g_{0\,(x,\eta,y,\xi)}}(y,\eta)$$

for $k\in\mathbb{Z}_+$ and $(x,\xi)\in\mathbb{R}^{2n}$. It follows from (A-2) and (20) that

(34) $$m_1(x,\eta)m_2(y,\xi)$$
$$\leq C^2 m(x,\xi)(1+g^\sigma_{2\,(x,\xi)}(0,\xi-\eta))^N(1+g^\sigma_{1\,(x,\xi)}(x-y,0))^N$$
$$\leq C' m(x,\xi)(1+g^A_{0\,(x,\xi,x,\xi)}(x-y,\xi-\eta))^{2N},$$

where $C' > 0$. From (31) and (34) there are $C > 0$ and N' such that

(35) $$M(x,\eta,y,\xi) \leq Cm(x,\xi)(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{N'}$$

for $(x,\xi),(y,\eta)\in\mathbb{R}^{2n}$. Let $(x,\xi)\in\mathbb{R}^{2n}$, and choose $(\hat{y}_\nu,\hat{\eta}_\nu)\in\mathbb{R}^{2n}$ so that $(x,\hat{\eta}_\nu,\hat{y}_\nu,\xi)\in U_\nu$ and

(36) $$d_\nu(x,\xi) \leq g^A_{0\,(x,,\hat{\eta}_\nu,\hat{y}_\nu,\xi)}(x-\hat{y}_\nu,\xi-\hat{\eta}_\nu) \leq [d_\nu(x,\xi)]+1.$$

Then, from (35) we have, with $C' > 0$,

(37) $$M(x,\eta,y,\xi) \leq C(m_1)C(m_2)M(x,,\hat{\eta}_\nu,\hat{y}_\nu,\xi)$$
$$\leq C' m(x,\xi)(1+d_\nu(x,\xi))^{N'} \quad \text{if } (x,\eta,y,\xi)\in U_\nu.$$

Therefore, from Lemma 5, (33) and (37) there is $k_0\in\mathbb{N}$ satisfying

(38) $$\sum_{\nu=1}^{\infty} |(\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,x,\xi)|$$

$$\leq Cm(x,\xi)\sup_{j\leq k_0}|u(x,\cdot,\cdot,\xi)|_j^{g_{0(x,\eta,y,\xi)}}(y,\eta)/M(x,\eta,y,\xi).$$

Let $B=\{v_j\}_{j=1,2,\cdots}$ be a bounded subset of $S(M,G)$. Then the Ascoli-Arzelà theorem implies that $v_j\to v$ in $C^\infty(\mathbb{R}^{4n})$ as $j\to\infty$ and $v\in S(M,G)$ if $v_j(X,Y)\to v(X,Y)$ as $j\to\infty$ for every $(X,Y)\in\mathbb{R}^{4n}$. It is obvious that

$$\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,y,\eta)=\sum_{\nu=1}^{\infty}\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,y,\eta)$$

for $u\in S(M,G)\cap C_0^\infty(\mathbb{R}^{4n})$. Assume that $v_j\in C_0^\infty(\mathbb{R}^{4n})$ and $v_j\to v$ in $C^\infty(\mathbb{R}^{4n})$. Note that $v\in S(M,G)$. Write $v_{j,\nu}=\Phi_\nu v_j$. By (38) with u_ν replaced by $v_{j,\nu}$ or its proof, for any $(x,\xi)\in\mathbb{R}^{2n}$ and $\varepsilon\geq 0$ there is $\nu_0\in\mathbb{N}$ such that

$$\Big|\sum_{\nu=\nu_0}^{\infty}(\exp(iA(D_\xi,D_y))v_{j,\nu})(x,\xi,x,\xi)\Big|<\varepsilon/3\quad(j=1,2,\cdots).$$

Since $D^\alpha v_j\to D^\alpha v$ uniformly on $\bigcup_{\nu=1}^{\nu_0-1}U_\nu$, from (33) there is $j_0\in\mathbb{N}$ such that

$$\Big|\sum_{\nu=1}^{\nu_0-1}(\exp(iA(D_\xi,D_y))(v_{j,\nu}-v_{j',\nu}))(x,\xi,x,\xi)\Big|<\varepsilon/3\quad\text{if } j,j'\geq j_0,$$

which gives

$$|(\exp(iA(D_\xi,D_y))v_j)(x,\xi,x,\xi)-(\exp(iA(D_\xi,D_y))v_{j'})(x,\xi,x,\xi)|<\varepsilon$$

if $j,j'\geq j_0$. So, for $(x,\xi)\in\mathbb{R}^{2n}$ $\{(\exp(iA(D_\xi,D_y))v_j)(x,\xi,x,\xi)\}_{j=1,2,\cdots}$ converges in $\mathbb{C}$. Therefore, we can define

$$(\exp(iA(D_\xi,D_y))v)(x,\xi,x,\xi)=\lim_{j\to\infty}(\exp(iA(D_\xi,D_y))v_j)(x,\xi,x,\xi)$$

for $(x,\xi)\in\mathbb{R}^{2n}$. Recall that $B=\{v_j\}_{j=1,2,\cdots}$ is a bounded subset of $S(M,G)$, and assume that $v\in S(M,G)$ and $v_j\to v$ in $C^\infty(\mathbb{R}^{4n})$ as $j\to\infty$. We put

$$v_j^k=\sum_{\nu=1}^{k}\Phi_\nu v_j\quad(k=1,2,\cdots).$$

Then $\{v_j^k\}_{j,k=1,2,\cdots}(\subset C_0^\infty(\mathbb{R}^{4n}))$ is bounded in $S(M,G)$ and $v_j^k\to v_j$ in $C^\infty(\mathbb{R}^{4n})$ as $k\to\infty$. Let $(x,\xi)\in\mathbb{R}^{2n}$ and $\varepsilon>0$. There is $K\in\mathbb{N}$ satisfying

$$|(\exp(iA(D_\xi,D_y))v_j)(x,\xi,x,\xi)-(\exp(iA(D_\xi,D_y))v_j^k)(x,\xi,x,\xi)|<\varepsilon/2$$

for $k \geq K$ and $j \in \mathbb{N}$. In particular,

$$|(\exp(iA(D_\xi, D_y))v_j)(x,\xi,x,\xi) - (\exp(iA(D_\xi, D_y))v_j^j)(x,\xi,x,\xi)| < \varepsilon/2$$

for $j \geq K$. It is obvious that $v_j^j \to v$ in $C^\infty(\mathbb{R}^{4n})$ as $j \to \infty$. Therefore, we have

$$(\exp(iA(D_\xi, D_y))v)(x,\xi,x,\xi) = \lim_{j\to\infty}(\exp(iA(D_\xi, D_y))v_j^j)(x,\xi,x,\xi),$$

which implies that for each $(x,\xi) \in \mathbb{R}^{2n}$ there is $j_0 \in \mathbb{N}$ satisfying

$$|(\exp(iA(D_\xi, D_y))v)(x,\xi,x,\xi) - (\exp(iA(D_\xi, D_y))v_j)(x,\xi,x,\xi)| < \varepsilon \quad \text{if } j \geq j_0.$$

So we have the following

Theorem 6 (Theorem 18.4.10 in [1]). *For each $(x,\xi) \in \mathbb{R}^{2n}$ the linear form $C_0^\infty(\mathbb{R}^{4n}) \ni u \mapsto (\exp(iA(D_\xi, D_y))u)(x,\xi,x,\xi) \in \mathbb{C}$ can be extended uniquely to a weakly continuous linear form, i.e.,*

$$(\exp(iA(D_\xi, D_y))v_j)(x,\xi,x,\xi) \to (\exp(iA(D_\xi, D_y))v)(x,\xi,x,\xi) \quad \text{as } j \to \infty$$

if $\{v_j\}_{j=1,2,\cdots}$ is bounded in $S(M,G)$ and $v_j \to v$ in $C^\infty(\mathbb{R}^{4n})$ as $j \to \infty$. Moreover, there are $k_0 \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned}|(\exp(iA(D_\xi, D_y))u)(x,\xi,x,\xi)| &\leq Cm(x,\xi) \\ &\times \sup_{j\leq k_0}\sup_{(y,\eta)} |u(x,\cdot,\cdot,\xi)|_j^{g_0(x,\eta,y,\xi)}(y,\eta)/M(x,\eta,y,\xi).\end{aligned}$$

Here k_0 and C depend only on the constants in (1), (2), (21) *and* (35).

Let $(x,\xi) \in \mathbb{R}^{2n}$ and $\nu \in \mathbb{N}$. From (18) it follows that for $p \in \mathbb{Z}_+$ there is $C_{p,R_0,R} > 0$ satisfying

$$\begin{aligned}(39)\qquad &|\langle D_{x,\xi}, t_1\rangle \cdots \langle D_{x,\xi}, t_k\rangle(\exp(iA(D_\xi, D_y))u_\nu)(x,\xi,x,\xi)| \\ &= |\exp(iA(D_\xi, D_y))\langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle \cdots \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu|_{y=x,\,\eta=\xi} \\ &\leq C_{p,R_0,R}(1 + \inf_{(x,\eta,y,\xi)\in U_\nu} g^A_{0(X_\nu,Y_\nu)}(x-y,\xi-\eta))^{-p/2} \\ &\quad \times \sup_{j\leq n+1+p}\sup_{(y,\eta)} |(\langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle \cdots \\ &\qquad\qquad \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu)(x,\cdot,\cdot,\xi)|_j^{g_0(X_\nu,Y_\nu)}(y,\eta),\end{aligned}$$

where $t_1, \cdots, t_k \in \mathbb{R}^{2n}$. Let $(\hat{y}_\nu, \hat{\eta}_\nu) \in \mathbb{R}^{2n}$ satisfy $(x, \hat{\eta}_\nu, \hat{y}_\nu, \xi) \in U_\nu$ and (36). Then by (21) we have

$$G_{(X_\nu,Y_\nu)}(t_l,t_l) \leq C_0 G_{(x,\hat{\eta}_\nu,\hat{y}_\nu,\xi)}(t_l,t_l)$$

$$\leq C_0 C(G) G_{(x,\xi,x,\xi)}(t_l,t_l)(1+g^A_{0(x,\hat\eta_\nu,\hat y_\nu,\xi)}(x-\hat y_\nu,\xi-\hat\eta_\nu))^{N(G)}$$
$$\leq C'(G) g_{(x,\xi)}(t_l)(1+d_\nu(x,\xi))^{N(G)} \quad (\ l=1,2,\cdots,k),$$

where $C'(G)=2^{N(G)+1}C_0C(G)$. So we have

$$|(\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu)(x,\cdot,\cdot,\xi)|_j^{g_{0(X_\nu,Y_\nu)}}(y,\eta)$$
$$= \sup_{s_1,\cdots,s_j\in\mathbb{R}^{2n}} |(\langle D_{y,\xi},s_1\rangle\cdots\langle D_{y,\xi},s_j\rangle\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle$$
$$\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu)(x,\eta,y,\xi)|\prod_{\mu=1}^j g_{0(X_\nu,Y_\nu)}(s_\mu)^{-1/2}$$
$$\leq C'(G)^{k/2}\prod_{l=1}^k g_{(x,\xi)}(t_l)^{1/2}(1+d_\nu(x,\xi))^{N(G)k/2}$$
$$\times \sup_{s_1,\cdots,s_j\in\mathbb{R}^{2n}} |(\langle D_{y,\xi},s_1\rangle\cdots\langle D_{y,\xi},s_j\rangle\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle$$
$$\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu)(x,\eta,y,\xi)|$$
$$\times\prod_{l=1}^k G_{(X_\nu,Y_\nu)}(t_l,t_l)^{-1/2}\prod_{\mu=1}^j g_{0(X_\nu,Y_\nu)}(s_\mu)^{-1/2}$$
$$\leq (C_0C'(G))^{k/2}C_0^{j/2}\prod_{l=1}^k g_{(x,\xi)}(t_l)^{1/2}(1+d_\nu(x,\xi))^{N(G)k/2}|u_\nu|^G_{j+k}(x,\eta,y,\xi).$$

This, together with (37) and (39), yields

$$\begin{aligned}(40)\qquad &|(\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,x,\xi)|_k^g\\ &\leq C_{k,p,R_0,R}(G)m(x,\xi)(1+d_\nu(x,\xi))^{-p/2+N'+N(G)k/2}\\ &\times\sup_{j\leq n+1+p+k}\sup_{(y,\eta)}|u_\nu|^G_j(x,\eta,y,\xi)/M(x,\eta,y,\xi),\end{aligned}$$

where $C_{k,p,R_0,R}>0$. Therefore, we have the following

Theorem 7 (Theorem 18.4.10′ in [1])**.** *For any $k\in\mathbb{Z}_+$ there are $k_0\in\mathbb{N}$ and $C_k>0$ such that*

$$|(\exp(iA(D_\xi,D_y))u)(x,\xi,x,\xi)|_k^g$$
$$\leq C_k m(x,\xi)\sup_{j\leq k_0}\sup_{(y,\eta)}|u|^G_j(x,\eta,y,\xi)/M(x,\eta,y,\xi)$$

for $u\in S(M,G)$ and $(x,\xi)\in\mathbb{R}^{2n}$. Here k_0 and C depend only on k and the constants in (1) – (3), (21) *and* (35)*. Moreover, the linear map: $S(M,G)\ni u\mapsto (\exp(iA(D_\xi,D_y))u)(x,\xi,x,\xi)\in S(m,g)$ is weakly continuous.*

Define

$$H(X,Y)=\Big\{\sup_{t\in\mathbb{R}^{2n}\setminus\{0\}}\frac{g_{0\,(X,Y)}(t)}{g^A_{0\,(X,Y)}(t)}\Big\}^{1/2}\quad (\,X,Y\in\mathbb{R}^{2n}).$$

Recall that $h(X)=H(X,X)$ ($X\in\mathbb{R}^{2n}$). Then $H(X,Y)$ is G continuous and

$$H(x,\eta,y,\xi)\le C(G)h(x,\xi)(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{N(G)}$$

for $(x,\xi),(y,\eta)\in\mathbb{R}^{2n}$. Indeed, by (21) and (31) we have

$$H(x,\eta,y,\xi)^2\le\sup_{t\in\mathbb{R}^{2n}\setminus\{0\}}\frac{C(G)g_{0\,(x,\xi,x,\xi)}(t)}{C(G)^{-1}g^A_{0\,(x,\xi,x,\xi)}(t)}(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{2N(G)}$$
$$=C(G)^2h(x,\xi)^2(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{2N(G)}.$$

Let $l\in\mathbb{Z}_+$, and put

$$R_l(x,\xi;u)=\sum_{\nu=1}^{\infty}\Big\{(\exp(iA(D_\xi,D_y))u_\nu)(x,\xi,x,\xi)$$
$$-\sum_{j<l}[(iA(D_\xi,D_y))^j u_\nu(x,\xi,y,\eta)]_{y=x,\,\eta=\xi}/j!\Big\}$$

for $u\in S(M,G)$. Note that $R_l(x,\xi)=R_l(x,\xi;a_1a_2)$, which is defined by (5). Suppose that $(x,\xi,x,\xi)\notin U'_\nu$. Then we have

$$g_{0\,(X_\nu,Y_\nu)}(x-y,\xi-\eta)=G_{(X_\nu,Y_\nu)}(0,\xi-\eta,x-y,0)\ge(c_0^{1/2}-R_0)^2>0,$$
$$(\,0<\,)\,c_1\equiv C_0^{-1}(c_0^{1/2}-R_0)^2\le g_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta)$$
$$\le H(x,\eta,y,\xi)^2g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta)$$
$$\le C(G)^2h(x,\xi)^2(1+g^A_{0\,(x,\eta,y,\xi)}(x-y,\xi-\eta))^{2N(G)+1}$$

if $(x,\eta,y,\xi)\in U_\nu$. Therefore, we have

(41) $$1\le(C(G)/\sqrt{c_1})h(x,\xi)(1+d_\nu(x,\xi))^{N(G)+1/2},$$

noting that $d_\nu(x,\xi)=\infty$ unless there is $(y^1,\eta^1)\in\mathbb{R}^{2n}$ satisfying $(x,\eta^1,y^1,\xi)\in U_\nu$. By (41) we have

(42) $$|(\exp(iA(D_\xi,D_y))\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)$$
$$-\sum_{j<l}(iA(D_\xi,D_y))^j$$

$$\times \langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle \cdots \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)/j!|_{y=x,\,\eta=\xi}$$
$$\leq C'_{l,k,p,R_0,R}(G)m(x,\xi)\prod_{\mu=1}^{k} g_{(x,\xi)}(t_\mu)^{1/2}h(x,\xi)^l$$
$$\times (1+d_\nu(x,\xi))^{-p/2+N'+N(G)(k/2+l)+l/2}$$
$$\times \sup_{j\leq n+1+p+k}\sup_{(y,\eta)} |u_\nu|_j^G(x,\eta,y,\xi)/M(x,\eta,y,\xi)$$

for $t_1,\cdots,t_k \in \mathbb{R}^{2n}$, applying the same argument as for (40), since $(iA(D_\xi,D_y))^j \times u_\nu(x,\xi,y,\eta)|_{y=x,\,\eta=\xi} = 0$. Next suppose that $(x,\xi,x,\xi) \in U'_\nu$. It follows from (6) that

$$(43)\quad |\exp(iA(D_\xi,D_y))\langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle \cdots \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)$$
$$-\sum_{j<l}(iA(D_\xi,D_y))^j$$
$$\times \langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle \cdots \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)/j!|_{y=x,\,\eta=\xi}$$
$$\leq C_l \sup_{j\leq n+1}\sup_{(y^1,\eta^1)} |((iA(D_\xi,D_y))^l \langle D_{x,\xi,y,\eta}, (t_1,t_1)\rangle$$
$$\cdots \langle D_{x,\xi,y,\eta}, (t_k,t_k)\rangle u_\nu)(x,\cdot,\cdot,\xi)|_j^{g_{0\,(x,\xi^1,y^1,\xi)}}(y^1,\xi^1).$$

Let $(x,\xi^1,y^1,\xi) \in U_\nu$. We can assume without loss of generality that $g_{0\,(x,\xi^1,y^1,\xi)}$ is equal to the square of Euclidean norm $|\cdot|$ of $\mathbb{R}^{2n}$, *i.e.*, $g_{0\,(x,\xi^1,y^1,\xi)}(X) = \sum_{j=1}^{2n} X_j^2$. Moreover, choosing an orthonormal basis of $\mathbb{R}^{2n}$ suitably, we may assume that

$$g_{0\,(x,\xi^1,y^1,\xi)}(X) = \sum_{j=1}^{2n} X_j^2 \quad \text{for } X \in \mathbb{R}^{2n},$$
$$A(D_\xi,D_y) = \sum_{j=1}^{2n} b_j D_{X_j}^2.$$

Then we have

$$A = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_{2n} \end{pmatrix},$$
$$g^A_{0\,(x,\xi^1,y^1,\xi)}(X) = \sum_{j=1}^{2n} b_j^{-2} X_j^2,$$
$$H(x,\xi^1,y^1,\xi) = \sup_{1\leq j\leq 2n} |b_j|.$$

Therefore, we have

$$(44)\qquad |((iA(D_\xi,D_y))^l\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle \cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu)(x,\cdot,\cdot,\xi)|_j^{g_{0\,(x,\xi^1,y^1,\xi)}}(y^1,\xi^1)$$
$$\le H(x,\xi^1,y^1,\xi)^l|u_\nu|^G_{j+2l+k}(x,\xi^1,y^1,\xi)\prod_{\mu=1}^k G_{(x,\xi^1,y^1,\xi)}(t_\mu,t_\mu)^{1/2}.$$

Since $H(x,\xi^1,y^1,\xi)\le C_0h(x,\xi)$, $G_{(x,\xi^1,y^1,\xi)}(t_\mu,t_\mu)\le 2C_0g_{(x,\xi)}(t_\mu)$ and $M(x,\xi^1,$ $y^1,\xi)\le C(M)m(x,\xi)$ if $(x,\xi^1,y^1,\xi)\in U_\nu$, from (43) and (44) we see that, with some $C_{l,k,R_0,R}>0$,

$$\begin{aligned}&|\exp(iA(D_\xi,D_y))\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)\\ &\quad-\sum_{j<l}(iA(D_\xi,D_y))^j\\ &\qquad\times\langle D_{x,\xi,y,\eta},(t_1,t_1)\rangle\cdots\langle D_{x,\xi,y,\eta},(t_k,t_k)\rangle u_\nu(x,\xi,y,\eta)/j!|_{y=x,\,\eta=\xi}\\ &\le C_{l,k,R_0,R}m(x,\xi)\prod_{\mu=1}^k g_{(x,\xi)}(t_\mu)^{1/2}h(x,\xi)^l\\ &\quad\times\sup_{j\le n+1+2l+k}\ \sup_{(y,\eta)}|u_\nu|_j^G(x,\eta,y,\xi)/M(x,\eta,y,\xi).\end{aligned}$$

Therefore, for any $l,k\in\mathbb{Z}_+$ there are $k_0\in\mathbb{N}$ and $C_{l,k}>0$ such that

$$\begin{aligned}&|R_l(\cdot,\cdot;u)|_k^g(x,\xi)\le C_{l,k}m(x,\xi)h(x,\xi)^l\\ &\quad\times\sup_{j\le k_0}\sup_{(y,\eta)}|u|_j^G(x,\eta,y,\xi)/M(x,\eta,y,\xi)\quad\text{for } u\in S(M,G).\end{aligned}$$

This, together with Theorem 7, proves Theorem 1.

Example 8. Let $\rho_j\in[0,1]$ and $\delta_j\in[0,1)$ ($j=1,2$) satisfy $\delta_2\le\rho_1$. Define

$$g_{j\,(x,\xi)}=\langle\xi\rangle^{2\delta_j}|dx|^2+\langle\xi\rangle^{-2\rho_j}|d\xi|^2\quad(j=1,2),$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. Then we have

$$\begin{aligned}&g^\sigma_{j\,(x,\xi)}=\langle\xi\rangle^{2\rho_j}|dx|^2+\langle\xi\rangle^{-2\delta_j}|d\xi|^2,\\ &g_{0\,(x,\xi,x,\xi)}=\langle\xi\rangle^{2\delta_2}|dx|^2+\langle\xi\rangle^{-2\rho_1}|d\xi|^2,\\ &g_{(x,\xi)}=(((\langle\xi\rangle^{2\delta_1}+\langle\xi\rangle^{2\delta_2})|dx|^2+(\langle\xi\rangle^{-2\rho_1}+\langle\xi\rangle^{-2\rho_2})|d\xi|^2)/2,\\ &g^A_{0\,(x,\xi,x,\xi)}=4(\langle\xi\rangle^{2\rho_1}|dx|^2+\langle\xi\rangle^{-2\delta_2}|d\xi|^2),\\ &h(x,\xi)=\langle\xi\rangle^{\delta_2-\rho_1}/2.\end{aligned}$$

Then g_j ($j = 1,2$) are σ temperate Riemannian metrics and the assumptions (A-1), (A-3) and (A-4) are satisfied. Let $\mu_j \in \mathbb{R}$ ($j = 1,2$), and let $a_j \in S^{\mu_j}_{\rho_j,\delta_j}$ $(= S(\langle\xi\rangle^{\mu_j}, g_j))$ ($j = 1,2$). Then the assumption (A-2) with $m_j(x,\xi) = \langle\xi\rangle^{\mu_j}$ is satisfied. Theorem 1 implies that

$$a_1(x,\xi) \circ a_2(x,\xi) - \sum_{|\alpha|<l} a_1^{(\alpha)}(x,\xi) a_{2(\alpha)}(x,\xi)/\alpha! \in S^{\mu_1+\mu_2-l(\rho_1-\delta_2)}_{\rho,\delta},$$

where $\rho = \min\{\rho_1, \rho_2\}$ and $\delta = \max\{\delta_1, \delta_2\}$. On the other hand, Theorem 18.5.5 of [1] implies that

$$(a_1 \# a_2)(x,\xi) - \sum_{|\alpha|+|\beta|<l} (-1)^{|\beta|} 2^{-|\alpha|-|\beta|} a^{(\alpha)}_{1(\beta)}(x,\xi) a^{(\beta)}_{2(\alpha)}(x,\xi)/(\alpha!\beta!)$$
$$\in S^{\mu_1+\mu_2-l\min\{\rho_2-\delta_1,\rho_1-\delta_2\}}_{\rho,\delta}.$$

If, for example, $\rho_2 = \delta_1$ and $\rho_1 > \delta_2$, then the classical calculus for pseudo-differential operators is better than the Weyl calculus in some sense.

References

[1] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.