Asymptotic expansions of the roots of the equations of pseudo-polynomials with a small parameter

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In the studies of partial differential operators we frequently need to know asymptotic expansions of the roots of the equations of pseudo-polynomials with a small parameter. So we shall give a basic result on it here.

Let U be a non-void open subset of $\mathbb{R}^n$, and let $n(j) \in \mathbb{Z} \cup \{\infty\}$ and $a_j(s,\eta) \in C^\infty((0,1) \times U)$ ($0 \le j \le m$) satisfy the following:

(i) If $n(j) < \infty$, then there are $a_{j,k}(\eta) \in \mathscr{A}(U)$ ($k \ge n(j)$) such that $a_{j,n(j)}(\eta) \not\equiv 0$ and

$$a_j(s,\eta) \sim \sum_{k=n(j)}^{\infty} a_{j,k}(\eta) s^k \quad \text{in } U \text{ as } s \downarrow 0 \tag{1}$$

Here (1) means that for any compact subset K of U and $N \in \mathbb{N}$ ($\equiv \{1,2,3,\cdots\}$) there is $C_{K,N} > 0$ such that

$$\left| a_j(s,\eta) - \sum_{k=n(j)}^{N-1+n(j)} a_{j,k}(\eta) s^k \right| \le C_{K,N} s^{N+n(j)} \quad \text{for } s \in (0,1/2] \text{ and } \eta \in K,$$

where $\mathscr{A}(U)$ denotes the set of all real analytic functions defined in U.

(ii) If $n(j) = \infty$, then $a_j(s,\eta) = O(s^\infty)$ in U as $s \downarrow 0$, *i.e.*, for any compact subset K of U and $N \in \mathbb{N}$ there is $C_{K,N} > 0$ such that

$$|a_j(s,\eta)| \le C_{K,N} s^N \quad \text{for } s \in (0,1/2] \text{ and } \eta \in K.$$

We note that $a_j(s,\eta) := a_j(s\eta)$ ($0 \le j \le m$) satisfy the above if U is star-shaped with respect to the origin and $a_j(\eta) \in C^\infty(U)$ ($0 \le j \le m$). We assume that

$$n(m) < \infty. \tag{A}$$

Let

$$p(t,s,\eta) := \sum_{j=0}^{m} a_j(s,\eta)t^j,$$

and put $U' := \{\eta \in U;\ a_{m,n(m)}(\eta) \neq 0\}$. Then for each compact subset K of U' there are $\delta_K \in (0,1)$ and $\tau_j(s,\eta)$ ($1 \le j \le m$) defined in $(0,\delta_K] \times K$ such that

$$p(t,s,\eta) = a_m(s,\eta)\prod_{j=1}^{m}(t-\tau_j(s,\eta)), \quad a_m(s,\eta) \neq 0$$

for $s \in (0,\delta_K]$ and $\eta \in K$. Note that $\{\tau_j(s,\eta)\}$ is uniquely determined as a multi-valued function: $(0,\delta_K] \times K \ni (s,\eta) \mapsto \{\tau_j(s,\eta)\} \in \mathscr{P}(\mathbb{C})$, where $\mathscr{P}(\mathbb{C})$ denotes the power set of $\mathbb{C}$.

Theorem. *Rearranging $\{\tau_j(s,\eta)\}$ if necessary, there are $N_0, L \in \mathbb{N}$, $\varphi_{(k)}(\eta) \in \mathscr{A}(U_{(k-1)})$ ($1 \le k \le N_0$) with $U_{(0)} \equiv U$ and $U_{(k)} \equiv \{\eta \in U_{(k-1)};\ \varphi_{(k)}(\eta) \neq 0\}$ ($1 \le k \le N_0$), $r \in \mathbb{N}$ with $1 \le r \le m$, $j_\mu \in \mathbb{N}$ ($\mu = 1,2,\cdots,r-1$) with $(0<) j_1 < j_2 < \cdots < j_{r-1} < m$ and $\mu(k,l) \in \mathbb{Z}$ and $\tau_{k,l}(\eta) \in \mathscr{A}(U_{(N_0)})$ ($1 \le k \le r$ and $l \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$) such that $U_{(1)} \subset U'$, $\varphi_{(k)}(\eta) \not\equiv 0$, $\mu(k,l) < \mu(k,l+1)$ ($l \in \mathbb{Z}_+$) and*

$$\tau_{j_{k-1}+i}(s,\eta) \sim \sum_{l=0}^{\infty} \tau_{k,l}(\eta) s^{\mu(k,l)/L} \quad \text{in } U_{(N_0)} \text{ as } s \downarrow 0$$

($1 \le k \le r$ and $1 \le i \le j_k - j_{k-1}$), where $j_0 = 0$ and $j_r = m$.

In the rest of this note we shall prove the theorem. In doing so, we need the following

Lemma. *Let $b_j(\eta) \in \mathscr{A}(U)$ ($0 \le j \le m$). Assume that $b_0(\eta) \not\equiv 0$, and put*

$$q(t,\eta) := b_0(\eta)t^m + b_1(\eta)t^{m-1} + \cdots + b_m(\eta).$$

Then there is $\varphi(\eta) \in \mathscr{A}(U)$ such that $\varphi(\eta) \not\equiv 0$ and $b_0(\eta) \neq 0$ and the multiplicities of the roots of $q(t,\eta) = 0$ in t are constant for $\eta \in \widetilde{U}$, where $\widetilde{U} := \{\eta \in U;\ \varphi(\eta) \neq 0\}$. Moreover, there are $r \in \mathbb{N}$ with $1 \le r \le m$, $\tau_k(\eta) \in \mathscr{A}(\widetilde{U})$ and $m_k \in \mathbb{N}$ ($1 \le k \le r$) such that

$$q(t,\eta) = b_0(\eta)\prod_{k=1}^{r}(t-\tau_k(\eta))^{m_k} \quad \text{for } \eta \in \widetilde{U},$$

$$\tau_j(\eta) \neq \tau_k(\eta) \quad \text{for } \eta \in \widetilde{U} \text{ if } j \neq k.$$

Proof. Note that $\mathscr{A}(U)$ is an integral domain, and denote by $\mathscr{K}$ the quotient field of $\mathscr{A}(U)$. We write

$$q(t,\eta) = c(\eta)q_1(t,\eta)^{k_1}\cdots q_\nu(t,\eta)^{k_\nu},$$

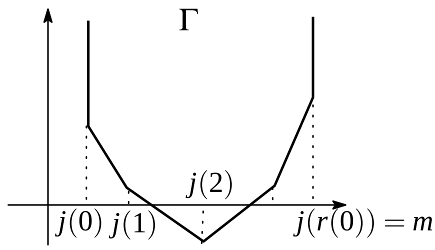
where $c(\eta) \in \mathscr{K}$, $q_j(t,\eta) \in \mathscr{K}[t]$ ($1 \le j \le \nu$), $\deg_t q_j(t,\eta) \ge 1$ and the $q_j(t,\eta)$ are mutually prime in $\mathscr{K}[t]$. Put

$$Q(t,\eta) := q_1(t,\eta)\cdots q_\nu(t,\eta),$$

and denote by $R(\eta)$ the discriminant of the equation $Q(t,\eta) = 0$ in t, *i.e.*, the resultant of $Q(t,\eta)$ and $(\partial Q/\partial t)(t,\eta)$ as polynomials of t. Then we have $R(\eta) \not\equiv 0$. Since $R(\eta) \in \mathscr{K}$, there is $\psi(\eta) \in \mathscr{A}(U)$ such that $\psi(\eta) \not\equiv 0$ and $\psi(\eta)R(\eta) \in \mathscr{A}(U)$. Putting $\varphi(\eta) := \psi(\eta)R(\eta)$ we can prove the lemma, since the roots of $Q(t,\eta) = 0$ in t are all simple if $\eta \in U$ and $R(\eta) \neq 0$. □

(I) Put

$$\Gamma := \mathrm{ch}\Big[\bigcup_{\substack{0\le j\le m\\ n(j)<\infty}} \{j\}\times[n(j),\infty)\Big],$$



where $\mathrm{ch}[A]$ denotes the convex hull of A. Γ is the Newton polygon of $p(t,s,\eta)$ with respect to (t,s). Later we shall also treat a slightly different kind of Newton polygons. Let $(j(k),\nu(k))$ ($k = 0,1,\cdots,r(0)$) be the vertexes of Γ arranged as follows;

$$0 (= j(-1)) \le j(0) < j(1) < \cdots < j(r(0)) = m.$$

Note that $\nu(k) = n(j(k))$ ($0 \le k \le r(0)$). We denote by $l(0), l(1), \cdots, l(r(0)+1)$ the sides of Γ, *i.e.*, $l(0)$ is the half-line connecting $(j(0),\infty)$ and $(j(0),\nu(0))$, $l(k)$ is the segment connecting $(j(k-1),\nu(k-1))$ and $(j(k),\nu(k))$ ($1 \le k \le r(0)$), and $l(r(0)+1)$ is the half-line connecting $(m,\nu(r(0)))$ and (m,∞). Denote by $-\kappa(k)$ the slope of $l(k)$ ($1 \le k \le r(0)$) and put $\kappa(0) = \infty$ and $\kappa(r(0)+1) = -\infty$. Moreover, we define

$$p_k(t,\eta) := \sum_{(j,n(j))\in l(k)} a_{j,n(j)}(\eta)t^{j-j(k-1)} \quad (1 \le k \le r(0)),$$

$$\psi_{(0)}(\eta) := \prod_{\substack{0\le k\le r(0)\\ j(k)\neq 0}} a_{j(k),\nu(k)}(\eta) \ (\not\equiv 0),$$

$$U'_{(1)} := \{\eta \in U;\ \psi_{(0)}(\eta) \neq 0\}.$$

(i) (the case $j(0) > 0$) Rearranging $\{\tau_j(s,\eta)\}$ if necessary, we have

(2) $$\tau_j(s,\eta) = O(s^\infty) \quad \text{in } U'_{(1)} \text{ as } s \downarrow 0 \ (1 \le j \le j(0)),$$

i.e., for any compact subset K of $U'_{(1)}$ there are $C_{K,N} > 0$ ($N \in \mathbb{N}$) such that

$$|\tau_j(s,\eta)| \le C_{K,N}s^N \quad \text{for } 1 \le j \le j(0), N \in \mathbb{N}, s \in (0,\delta_K] \text{ and } \eta \in K.$$

Indeed, choose $N \in \mathbb{N}$ so that $N > \kappa(1)$, and put

$$g(X,s,\eta) := a_{j(0),\nu(0)}(\eta)X^{j(0)},$$
$$f_N(X,s,\eta) := s^{-Nj(0)-\nu(0)}p(s^N X,s,\eta).$$

Since $\min_{j(0)<j\le m} N(j-j(0)) + n(j) - \nu(0) > 0$, for any compact subset K of $U'_{(1)}$ there is $C_{K,N} > 0$ such that

$$|f_N(X,s,\eta) - g(X,s,\eta)| \le C_{K,N}s$$

if $X \in \mathbb{C}$, $|X| = 1$, $0 < s \le 1/2$ and $\eta \in K$. So there is $\delta_{K,N} > 0$ such that $\delta_{K,N} \le 1/2$ and

$$|f_N(X,s,\eta) - g(X,s,\eta)| < |g(X,s,\eta)|$$

if $X \in \mathbb{C}$, $|X| = 1$, $0 < s \le \delta_{K,N}$ and $\eta \in K$. Therefore, Rouché's theorem implies that there are exactly $j(0)$ roots (counted with multiplicities) of the equation $p(s^N X,s,\eta) = 0$ with respect to X in the set $\{X \in \mathbb{C};\ |X| < 1\}$ if $0 < s \le \delta_{K,N}$ and $\eta \in K$. This proves (2).

(ii) Let $1 \le k \le r(0)$, and apply Lemma to $p_k(t,\eta)$. Then there are $s(k) \in \mathbb{N}$, $R_k^{(0)}(\eta) \in \mathscr{A}(U)$ and $\tau_{(k,j)}^{(0)}(\eta) \in \mathscr{A}(U_{(1)})$ and $m(k,j) \in \mathbb{N}$ ($1 \le j \le s(k)$) such that

(3) $$p_k(t,\eta) = a_{j(k),\nu(k)}(\eta)\prod_{j=1}^{s(k)}(t - \tau_{(k,j)}^{(0)}(\eta))^{m(k,j)} \quad \text{in } U_{(1)},$$
$$\tau_{(k,i)}^{(0)}(\eta) \ne \tau_{(k,j)}^{(0)}(\eta) \quad \text{for } \eta \in U_{(1)} \text{ if } i \ne j,$$

where

$$\varphi_{(1)}(\eta) := \psi_{(0)}(\eta)\prod_{\mu=1}^{r(0)} R_\mu^{(0)}(\eta),$$
$$U_{(1)} := \{\eta \in U;\ \varphi_{(1)}(\eta) \ne 0\}.$$

Note that $\sum_{j=1}^{s(k)} m(k,j) = j(k) - j(k-1)$.

(iii) Put

$$A_1 := \{(k,j) \in \mathbb{N}^2;\ 1 \le k \le r(0),\ 1 \le j \le s(k)\}.$$

It is obvious that $(1,1) \in A_1$ and $j(0)+\sum_{\alpha\in A_1} m(\alpha) = m$. For $\alpha = (k,j) \in A_1$ we put

$$\kappa(\alpha) := \kappa(k), \quad j(\alpha) := j(k) \quad \text{and} \quad \nu(\alpha) := \nu(k).$$

Moreover, we put

$$\mathscr{B}(s,\eta) := \{\tau_1(s,\eta),\cdots,\tau_m(s,\eta)\} \quad (\text{ counting multiplicities}),$$

$$B_0^M(s;\varepsilon) := \begin{cases} \{\lambda \in \mathbb{C};\ |\lambda| < s^{\kappa(1)+M}\varepsilon\} & \text{if } j(0) > 0, \\ \emptyset \quad \text{if } j(0) = 0, \end{cases}$$

$$B_\alpha(s,\eta;\varepsilon) := \{\lambda \in \mathbb{C};\ |s^{\kappa(\alpha)}\tau_\alpha^{(0)}(\eta) - \lambda| < s^{\kappa(\alpha)}\varepsilon\},$$

where $(s,\eta) \in \bigcup_{K\Subset U'}(0,\delta_K]\times K$, $M\in\mathbb{N}$, $\varepsilon > 0$ and $\alpha \in A_1$. Here, for two subsets A and B of $\mathbb{R}^n$ $A \Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\overset{\circ}{B}$ of B. Now we shall prove that the following proposition P(0) is valid:

P(0):

$(1)_0$ For any $K \Subset U_{(1)}$ there are $\varepsilon_K > 0$ and $s_K > 0$ such that the sets $B_0^1(s;\varepsilon_K)$ and $B_\alpha(s,\eta;\varepsilon_K)$ ($\alpha \in A_1$) are mutually disjoint for $\eta \in K$ and $s \in (0,s_K]$.

$(2)_0$ For any $K \Subset U_{(1)}$, $M \in \mathbb{N}$ and $\varepsilon \in (0,\varepsilon_K]$ there is $s_{K,\varepsilon,M} \in (0,s_K]$ such that $B_0^M(s;\varepsilon)$ exactly contains $j(0)$ elements of $\mathscr{B}(s,\eta)$ (counted with multiplicities) and $B_\alpha(s,\eta;\varepsilon)$ exactly contains $m(\alpha)$ elements of $\mathscr{B}(s,\eta)$ (counted with multiplicities) ($\alpha \in A_1$) for $\eta \in K$ and $s \in (0,s_{K,\varepsilon,M}]$.

$(3)_0$ $\quad j(0)+\sum_{\alpha\in A_1} m(\alpha) = m.$

Let $K \Subset U_{(1)}$. By the definition of $\{\tau_\alpha^{(0)}(\eta)\}$ we have $\tau_\alpha^{(0)}(\eta) \neq 0$ for $\eta \in K$. Since $\kappa(1) > \kappa(2) > \cdots > \kappa(r(0))$, the assertion $(1)_0$ of P(0) easily follows. In (i) we proved the assertion $(2)_0$ for $B_0^M(s;\varepsilon)$. Let $\alpha = (k,j) \in A_1$, and define

$$f(X,s,\eta) := s^{-\kappa(\alpha)j(\alpha)-\nu(\alpha)}p(s^{\kappa(\alpha)}(\tau_\alpha^{(0)}(\eta)+X),s,\eta)$$

for $(X,s,\eta) \in \mathbb{C}\times(0,1)\times U_{(1)}$. Then we have

$$f(X,s,\eta) = p_k(\tau_\alpha^{(0)}(\eta)+X,\eta)(\tau_\alpha^{(0)}(\eta)+X)^{j(k-1)} + s^{1/L(1)}q_\alpha(X,s,\eta),$$

where $L(1)$ is the smallest positive integer satisfying $L(1)\kappa(\beta) \in \mathbb{Z}$ ($\beta \in A_1$) and $q_\alpha(X,s,\eta)$ is a polynomial of X of degree m whose coefficients are in $C([0,1)\times U_{(1)})$. From (3) we have

$$p_k(\tau_\alpha^{(0)}(\eta)+X,\eta) = a_{j(k),\nu(k)}(\eta)X^{m(\alpha)} \prod_{\substack{1\le\mu\le s(k)\\ \mu\neq j}} (X+\tau_\alpha^{(0)}(\eta)-\tau_{(k,\mu)}^{(0)}(\eta))^{m(k,\mu)}.$$

This gives

$$f(X,s,\eta) = g(X,\eta) + X^{m(\alpha)+1}h(X,\eta) + s^{1/L(1)}q_\alpha(X,s,\eta),$$

where

$$g(X,\eta) = a_{j(k),\nu(k)}(\eta)\tau_\alpha^{(0)}(\eta)^{j(k-1)}X^{m(\alpha)}\prod_{\substack{1\le\mu\le s(k)\\ \mu\neq j}}(\tau_\alpha^{(0)}(\eta)-\tau_{(k,\mu)}^{(0)}(\eta))^{m(k,\mu)}$$

and $h(X,\eta)$ is a polynomial of X of degree $(m-m(\alpha)-1)$ whose coefficients are in $C(U_{(1)})$. Therefore, there is $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$ there is $\delta_{K,\varepsilon}>0$ satisfying

$$|f(X,s,\eta)-g(X,s,\eta)| < |g(X,s,\eta)|$$

if $X\in\mathbb{C}$, $|X|=\varepsilon$, $0<s\le\delta_{K,\varepsilon}$ and $\eta\in K$. It follows from Rouché's theorem that the equation $p(s^{\kappa(\alpha)}(\tau_\alpha^{(0)}(\eta)+X),s,\eta)=0$ in X has just $m(\alpha)$ roots inside $\{X\in\mathbb{C};\ |X|<\varepsilon\}$ (counted with multiplicities). Define

$$p^\alpha(t,s,\eta) := s^{-\kappa(\alpha)j(\alpha)-\nu(\alpha)}p(s^{\kappa(\alpha)}(\tau_\alpha^{(0)}(\eta)+t),s,\eta)(=f(t,s,\eta)).$$

Let us repeat the same argument, replacing $p(t,s,\eta)$ with $p^\alpha(t,s,\eta)$ and so on.

(II) Let $1\le\mu\le N$, and assume that $j(\alpha_{\mu-1};0)\in\mathbb{Z}_+$ and $\kappa(\alpha_{\mu-1};1)>0$ for $\alpha_{\mu-1}\in A_{\mu-1}$, A_μ ($\subset\mathbb{N}^{2\mu}$), $U_{(\mu)}$ ($\subset U_{(\mu-1)}$), $j(\alpha_\mu)\in\mathbb{Z}_+$, $\nu(\alpha_\mu)\in\mathbb{Z}_+$ and $\kappa(\alpha_\mu)>0$ for $\alpha_\mu\in A_\mu$, $L(\mu)\in\mathbb{N}$, $\tau_\alpha^{\alpha_{\mu-1}}(\eta)\in\mathscr{A}(U_{(\mu)})$ and $m(\alpha_{\mu-1},\alpha)\in\mathbb{N}$ for $(\alpha_{\mu-1},\alpha)\in A_\mu$ are determined so that $j(\alpha_{\mu-1};0)=j(0)$, $\kappa(\alpha_{\mu-1};1)=\kappa(1)$ and $\tau_\alpha^{\alpha_{\mu-1}}(\eta)=\tau_\alpha^{(0)}(\eta)$ and $m(\alpha_{\mu-1},\alpha)=m(\alpha)$ for $\alpha\in A_1$ when $\mu=1$, $L(\mu)$ is divisible by $L(\mu-1)$, $L(\mu)\kappa(\alpha_\mu)/L(\mu-1)\in\mathbb{Z}$ for $\alpha_\mu\in A_\mu$ and the proposition P$(\mu-1)$ below is valid, where $L(0)=1$. Put

$$B^M_{\alpha_{\mu-1},0}(s,\eta;\varepsilon) := \begin{cases} \{\lambda\in\mathbb{C};\ \left|\sum_{\nu=1}^{\mu-1}s^{\tilde\kappa(\alpha_\nu)}\tau_{\alpha^{(\nu)}}^{\alpha_{\nu-1}}(\eta)-\lambda\right| \\ \qquad < s^{\tilde\kappa(\alpha_{\mu-1})+(\kappa(\alpha_{\mu-1};1)+M)/L(\mu-1)}\varepsilon\} \\ \qquad\qquad \text{if } j(\alpha_{\mu-1};0)>0, \\ \emptyset \quad \text{if } j(\alpha_{\mu-1};0)=0, \end{cases}$$

for $\mu>1$, $\alpha_\mu=(\alpha^{(1)},\alpha^{(2)},\cdots,\alpha^{(\mu)})\in A_\mu$, $M\in\mathbb{N}$, $s>0$, $\eta\in U_{(\mu)}$ and $\varepsilon>0$, where $\alpha_\nu=(\alpha^{(1)},\cdots,\alpha^{(\nu)})$ ($1\le\nu\le\mu$) and $\tilde\kappa(\alpha_\mu)=\sum_{\nu=1}^{\mu}\kappa(\alpha_\nu)/L(\nu-1)$ for $\alpha_\mu=(\alpha^{(1)},\cdots,\alpha^{(\mu)})\in A_\mu$. For $\mu=1$ we put $B^M_{\alpha_{\mu-1},0}(s,\eta;\varepsilon)=B^M_0(s;\varepsilon)$. Define

$$B_{\alpha_\mu}(s,\eta;\varepsilon) := \{\lambda\in\mathbb{C};\ \left|\sum_{\nu=1}^{\mu}s^{\tilde\kappa(\alpha_\nu)}\tau_{\alpha^{(\nu)}}^{\alpha_{\nu-1}}(\eta)-\lambda\right| < s^{\tilde\kappa(\alpha_\mu)}\varepsilon\}$$

for $\alpha_\mu = (\alpha^{(1)}, \cdots, \alpha^{(\mu)}) \in A_\mu$. We assume that the following proposition P($\mu - 1$) is valid:

P($\mu - 1$):

$(1)_{\mu-1}$ For any $K \Subset U_{(\mu)}$ there are $\varepsilon_K > 0$ and $s_K > 0$ such that the sets $B^1_{\alpha_\nu,0}(s, \eta; \varepsilon_K)$ and $B_{\alpha_\mu}(s, \eta; \varepsilon_K)$ ($\alpha_\mu = (\alpha^{(1)}, \cdots, \alpha^{(\mu)}) \in A_\mu$, $1 \le \nu \le \mu$ and $\alpha_\nu = (\alpha^{(1)}, \cdots, \alpha^{(\nu)})$) are mutually disjoint for $\eta \in K$ and $s \in (0, s_K]$.

$(2)_{\mu-1}$ For any $K \Subset U_{(\mu)}$, $M \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_K]$ there is $s_{K,\varepsilon,M} \in (0, s_K]$ such that $B^M_{\alpha_\nu,0}(s, \eta; \varepsilon)$ exactly contains $j(\alpha_\nu; 0)$ elements of $\mathscr{B}(s, \eta)$ (counted with multiplicities) and $B_{\alpha_\mu}(s, \eta; \varepsilon)$ exactly contains $m(\alpha_\mu)$ elements of $\mathscr{B}(s, \eta)$ (counted with multiplicities) for $\eta \in K$ and $s \in (0, s_{K,\varepsilon,M}]$, where $\alpha_\mu = (\alpha^{(1)}, \cdots, \alpha^{(\mu)}) \in A_\mu$, $0 \le \nu \le \mu - 1$ and $\alpha_\nu = (\alpha^{(1)}, \cdots, \alpha^{(\nu)})$.

$(3)_{\mu-1}$ $j(0) + \sum_{\alpha \in A_1} j(\alpha; 0) + \cdots + \sum_{\alpha_{\mu-1} \in A_{\mu-1}} j(\alpha_{\mu-1}; 0) + \sum_{\alpha_\mu \in A_\mu} m(\alpha_\mu) = m$.

For $\alpha_N = (\alpha^{(1)}, \cdots, \alpha^{(N)}) \in A_N$ we define

$$p^{\alpha_N}(t, s, \eta) := s^{-\sigma(\alpha_N)} p\Big(\sum_{\mu=1}^{N} s^{\tilde{\kappa}(\alpha_\mu)} \tau^{\alpha_{\mu-1}}_{\alpha^{(\mu)}}(\eta) + s^{\tilde{\kappa}(\alpha_N)} t, s, \eta \Big),$$

where $\alpha_\mu = (\alpha^{(1)}, \cdots, \alpha^{(\mu)}) \in A_\mu$ ($1 \le \mu \le N - 1$) and

$$\sigma(\alpha_N) := \sum_{\mu=1}^{N} (\kappa(\alpha_\mu) j(\alpha_\mu) + \nu(\alpha_\mu))/L(\mu - 1).$$

Write

$$p^{\alpha_N}(t, s, \eta) = \sum_{j=0}^{m} a^{\alpha_N}_j(s, \eta) t^j.$$

Then there are $n(\alpha_N; j) \in \mathbb{Z}_+ \cup \{\infty\}$ and $a^{\alpha_N}_{j,k}(\eta) \in \mathscr{A}(U_{(N)})$ ($0 \le j \le m$, $k \ge n(\alpha_N; j)$) such that $a^{\alpha_N}_{j,n(\alpha_N;j)}(\eta) \not\equiv 0$ and

$$a^{\alpha_N}_j(s, \eta) \sim \sum_{k=n(\alpha_N;j)}^{\infty} a^{\alpha_N}_{j,k}(\eta) s^{k/L(N)} \quad \text{in } U_{(N)} \text{ as } s \downarrow 0,$$

if $n(\alpha_N; j) < \infty$, and

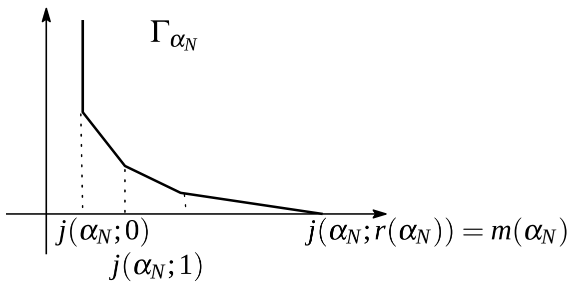
$$a^{\alpha_N}_j(s, \eta) = O(s^\infty) \quad \text{in } U_{(N)} \text{ as } s \downarrow 0,$$

if $n(\alpha_N; j) = \infty$. Put

$$\Gamma_{\alpha_N} := \mathrm{ch}\Big[\bigcup_{\substack{0 \le j \le m \\ n(\alpha_N;j) < \infty}} \{(j, n(\alpha_N; j))\} + (\overline{\mathbb{R}^n_+})^2 \Big],$$

where $\mathbb{R}_+ = \{\lambda \in \mathbb{R};\ \lambda > 0\}$. Let $(j(\alpha_N;k), \nu(\alpha_N;k)) \in (\mathbb{Z}_+)^2$ ($0 \le k \le r(\alpha_N)$) be the vertexes of Γ_{α_N} arranged as follows;

$$0 \le j(\alpha_N;0) < j(\alpha_N;1) < \cdots < j(\alpha_N;r(\alpha_N)) = m(\alpha_N).$$



We denote by $l(\alpha_N;k)$ the segment connecting $(j(\alpha_N;k-1), \nu(\alpha_N;k-1))$ and $(j(\alpha_N;k), \nu(\alpha_N;k))$ ($1 \le k \le r(\alpha_N)$). Denote by $-\kappa(\alpha_N;k)$ the slope of $l(\alpha_N;k)$ ($1 \le k \le r(\alpha_N)$). We note that $\kappa(\alpha_N;k) > 0$ ($1 \le k \le r(\alpha_N)$). Define

$$\begin{aligned}
&p_k^{\alpha_N}(t,\eta) := \sum_{(j,n(\alpha_N;j)) \in l(\alpha_N;k)} a^{\alpha_N}_{j,n(\alpha_N;j)}(\eta) t^{j-j(\alpha_N;k-1)} \quad (1 \le k \le r(\alpha_N)),\\
&\psi_{(N)}(\eta) := \prod_{\beta_N \in A_N} \prod_{\substack{0 \le k \le r(\beta_N)-1 \\ j(\beta_N;k) \neq 0}} a^{\alpha_N}_{j(\beta_N;k),\nu(\beta_N;k)}(\eta) \ (\not\equiv 0),\\
&U'_{(N+1)} := \{\eta \in U_{(N)};\ \psi_{(N)}(\eta) \neq 0\}.
\end{aligned}$$

(i) (the case $j(\alpha_N;0) > 0$) For any $M \in \mathbb{N}$ we can write

$$\begin{aligned}
&p^{\alpha_N}(s^{(\kappa(\alpha_N;1)+M)/L(N)}X, s, \eta)\\
&= s^{\{(\kappa(\alpha_N;1)+M)j(\alpha_N;0)+\nu(\alpha_N;0)\}/L(N)}\\
&\times \{a^{\alpha_N}_{j(\alpha_N;0),\nu(\alpha_N;0)}(\eta) X^{j(\alpha_N;0)} + s^{1/L(N+1)} q^M_{\alpha_N,0}(X,s,\eta)\}.
\end{aligned}$$

Here $L(N+1) \in \mathbb{N}$ is the smallest positive integer such that $L(N+1)$ is divisible by $L(N)$ and $L(N+1)\kappa(\beta_N;k)/L(N) \in \mathbb{Z}$ for any $\beta_N \in A_N$ and $1 \le k \le r(\beta_N)$, and $q^M_{\alpha_N,0}(X,s,\eta)$ is a polynomial of X of degree m whose coefficients are in $C([0,1) \times U_{(N)})$. From the same argument as in (I)(i) and Rouché's theorem it follows that for any $K \Subset U_{(N)}$ there is $\varepsilon_K > 0$ such that for any $\varepsilon \in (0, \varepsilon_K]$ and $M \in \mathbb{N}$ there is $s_{K,\varepsilon,M} > 0$ satisfying the following:

> $B^M_{\alpha_N,0}(s,\eta;\varepsilon)$ contains exactly $j(\alpha_N;0)$ elements of $\mathscr{B}(s,\eta)$ (counted with multiplicities) for $\eta \in K$ and $s \in (0, s_{K,\varepsilon,M}]$.

Let $1 \le k \le r(\alpha_N)$, and apply Lemma to the equation $p_k^{\alpha_N}(t,\eta) = 0$ in t. Then there are $R_k^{\alpha_N}(\eta) \in \mathscr{A}(U_{(N)})$, $s(\alpha_N;k) \in \mathbb{N}$, and $\tau^{\alpha_N}_{(k,j)}(\eta) \in \mathscr{A}(U_{(N+1)})$

and $m(\alpha_N,(k,j))\in\mathbb{N}$ ($1\le j\le s(\alpha_N;k)$) such that $a^{\alpha_N}_{j(\alpha_N;k),\nu(\alpha_N;k)}(\eta)\neq 0$ for $\eta\in U_{(N+1)}$ and

$$p^{\alpha_N}_k(t,\eta)=a^{\alpha_N}_{j(\alpha_N;k),\nu(\alpha_N;k)}(\eta)\prod_{j=1}^{s(\alpha_N;k)}(t-\tau^{\alpha_N}_{(k,j)}(\eta))^{m(\alpha_N,(k,j))}\quad \text{in } U_{(N+1)},$$
$$\tau^{\alpha_N}_{(k,i)}(\eta)\neq\tau^{\alpha_N}_{(k,j)}(\eta)\quad \text{for } \eta\in U_{(N+1)} \text{ if } i\neq j,$$

where

$$\varphi_{(N+1)}(\eta):=\psi_N(\eta)\prod_{\beta_N\in A_N}\prod_{\mu=1}^{r(\beta_N)}R^{\beta_N}_\mu(\eta),$$
$$U_{(N+1)}:=\{\eta\in U_{(N)};\ \varphi_{(N+1)}(\eta)\neq 0\}.$$

Put

$$A_{N+1}:=\{(\alpha_N,(k,j))\in\mathbb{N}^{2(N+1)};\ \alpha_N\in A_N,\ 1\le k\le r(\alpha_N) \text{ and } 1\le j\le s(\alpha_N;k)\}.$$

Since $(\alpha_N,(1,1))\in A_{N+1}$ if $\alpha_N\in A_N$, we have

$$\#A_N\le\#A_{N+1}\le m, \tag{4}$$

where $\#A_N$ denotes the number of the elements of A_N. Moreover, it is obvious that

$$r(\alpha_N)=s(\alpha_N;1)=1\quad \text{for any } \alpha_N\in A_N\quad \text{if } \#A_N=\#A_{N+1}. \tag{5}$$

Let $\alpha_N\in A_N$ and $\alpha^{(N+1)}=(k,j)$, and assume that $\alpha_{N+1}\equiv(\alpha_N,\alpha^{(N+1)})\in A_{N+1}$. Write

$$\kappa(\alpha_{N+1}):=\kappa(\alpha_N;k),\quad j(\alpha_{N+1}):=j(\alpha_N;k),\quad \nu(\alpha_{N+1}):=\nu(\alpha_N;k).$$

We have

$$\begin{aligned}
&s^{-\sigma(\alpha_{N+1})}p\Big(\sum_{\mu=1}^{N+1}s^{\tilde\kappa(\alpha_\mu)}\tau^{\alpha_{\mu-1}}_{\alpha^{(\mu)}}(\eta)+s^{\tilde\kappa(\alpha_{N+1})}X,s,\eta\Big)\\
&=s^{-(\kappa(\alpha_N;k)j(\alpha_N;k)+\nu(\alpha_N;k))/L(N)}p^{\alpha_N}(s^{\kappa(\alpha_N;k)/L(N)}(\tau^{\alpha_N}_{\alpha^{(N+1)}}(\eta)+X),s,\eta)\\
&=p^{\alpha_N}_k(\tau^{\alpha_N}_{\alpha^{(N+1)}}(\eta)+X,\eta)(\tau^{\alpha_N}_{\alpha^{(N+1)}}(\eta)+X)^{j(\alpha_N;k-1)}+s^{1/L(N+1)}q_{\alpha_{N+1}}(X,s,\eta)\\
&=a^{\alpha_N}_{j(\alpha_N;k),\nu(\alpha_N;k)}(\eta)X^{m(\alpha_{N+1})}(\tau^{\alpha_N}_{\alpha^{(N+1)}}(\eta)+X)^{j(\alpha_N;k-1)}\\
&\times\prod_{\substack{1\le\mu\le s(\alpha_N;k)\\ \mu\neq j}}(X+\tau^{\alpha_N}_{(k,j)}(\eta)-\tau^{\alpha_N}_{(k,\mu)}(\eta))^{m(\alpha_N,(k,\mu))}+s^{1/L(N+1)}q_{\alpha_{N+1}}(X,s,\eta),
\end{aligned}$$

where $q_{\alpha_{N+1}}(X,s,\eta)$ is a polynomial of X of degree m whose coefficients are in $C([0,1)\times U_{(N+1)})$. By the same argument as in (I)(ii) and Rouché's theorem we can prove that for any $K \Subset U_{(N+1)}$ and $\varepsilon \in (0,\varepsilon_K]$ there is $s_{K,\varepsilon} > 0$ such that $B_{\alpha_{N+1}}(s,\eta;\varepsilon)$ contains exactly $m(\alpha_{N+1})$ elements of $\mathscr{B}(s,\eta)$ for $\eta \in K$ and $s \in (0,s_{K,\varepsilon}]$, modifying ε_K if necessary. This implies that the proposition P(N) is valid for any $N \in \mathbb{Z}_+$. From (4), (5) and $(3)_N$ in P(N) it follows that there is $N_0 \in \mathbb{N}$ such that

$$\#A_N = \#A_{N_0}, \quad j(\alpha_N;0) = 0, \quad r(\alpha_N) = s(\alpha_N;1) = 1 \quad \text{for } N \geq N_0.$$

Note that $\psi_N(\eta) \equiv 1$ and $R_1^{\alpha_N}(\eta) \neq 0$ in $U_{(N)}$ for $\alpha_N \in A_N$ if $N \geq N_0$. Moreover, we have $L(N) = L(N_0)$ for $N \geq N_0$. Indeed, we have

$$\begin{aligned} &p_1^{\alpha_N}(t,\eta) = a^{\alpha_N}_{j(\alpha_N;1),\nu(\alpha_N;1)}(\eta)(t-\tau^{\alpha_N}_{(1,1)}(\eta))^{m(\alpha_N,(1,1))}, \\ &A_{N+1} = \{(\alpha_N,(1,1)) \in \mathbb{N}^{2(N+1)};\ \alpha_N \in A_N\} \end{aligned}$$

for $N \geq N_0$. Therefore, we have

$$\begin{aligned} &(j,n(\alpha_N;j)) \in l(\alpha_N;1) \quad (\,0 \leq j \leq j(\alpha_N;1)(\equiv m(\alpha_N,(1,1)))\,), \\ &\kappa(\alpha_N;1) = n(\alpha_N;0) - n(\alpha_N;1) = \cdots = n(\alpha_N;j(\alpha_N;1)-1) - n(\alpha_N;j(\alpha_N;1)) \\ &\in \mathbb{N} \end{aligned}$$

for $N \geq N_0$, which yields $L(N+1) = L(N)$ for $N \geq N_0$. This, together with P(N), proves the theorem.