On the norms of a symmetric multilinear form

T. Muramatu and S. Wakabayashi

April 22, 2009*

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let H be a pre-Hilbert space over $\mathbb{K}$ with inner product $(\cdot,\cdot)_H$, and let $f : H \times \cdots \times H \to \mathbb{C}$ be a symmetric n-linear form on H. Define

$$\begin{aligned} &\|f\|_1 = \sup\{|f(x,\cdots,x)|;\ x \in H \text{ and } \|x\| = 1\}, \\ &\|f\|_2 = \sup\{|f(x_1,\cdots,x_n)|;\ x_j \in H \text{ and } \|x_j\| = 1\ (\ 1 \le j \le n)\}, \end{aligned}$$

where $\|x\| = \sqrt{(x,x)_H}$. By definition it is obvious that $\|f\|_1 \le \|f\|_2$.

Theorem. *The two norms $\|f\|_1$ and $\|f\|_2$ coincide, i.e., $\|f\|_1 = \|f\|_2$.*

Remark. We thought the theorem must be well-known. However, we could not find any literature, and we gave a proof of the theorem (in 1992). Now we guess this theorem is not necessarily familiar among mathematicians. So we give its proof here.

To prove the theorem we need the following

Lemma. *Let $k \in \mathbb{N}$, and let g be a symmetric $2k$-linear form on H. Then we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g_{2j}(x+y, x-y) = 2^{2k} g_k(x,y) \quad \textit{for } x, y \in H, \tag{1}$$

where for $0 \le i \le 2k$

$$g_i(x,y) = g(x_1,\cdots,x_{2k})|_{x_j = x\ (1 \le j \le i),\ x_h = y\ (i+1 \le h \le 2k)}.$$

*In simplifying our original proof and writing the first version I made mistakes. As Prof. Lerner told me that it is not understandable, I noticed that its proof is incomplete, and correct the first version.

Proof. It is obvious that (1) is valid if $k=1$. Let $\ell\in\mathbb{N}$, and suppose that (1) is valid for $k\le\ell$. Then, for $k=\ell+1$ we have

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_{2j}(x+y,x-y)\\
&=\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k-1}{j}g_{2j}(x+y,x-y)+\sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1}g_{2j}(x+y,x-y)\\
&=-\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}g_{2j}(x+y,x-y)\\
&\quad+\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}g_{2j+2}(x+y,x-y),
\end{aligned}$$

since $\binom{k}{j}=\binom{k-1}{j}+\binom{k-1}{j-1}$ ($1\le j\le k-1$). By assumption we have

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}g_{2j}(x+y,x-y)\\
&=-2^{2(k-1)}\tilde{g}_{k-1}(x,y;x-y,x-y)+2^{2(k-1)}\tilde{g}_{k-1}(x,y;x+y,x+y)\\
&=2^{2k}g_k(x,y),
\end{aligned}$$

where

$$\tilde{g}_i(x,y;u,v)=g(x_1,\cdots,x_{2k})|_{x_j=x\ (1\le j\le i),\ x_h=y\ (i+1\le h\le 2k-2),\ x_{2k-1}=u,\ x_{2k}=v},$$

since $g_{2j}(x+y,x-y)=\tilde{g}_{2j}(x+y,x-y;x-y,x-y)$, $g_{2j+2}(x+y,x-y)=\tilde{g}_{2j}(x+y,x-y;x+y,x+y)$, $g(x_1,\cdots,x_{2\ell},u,u)$ is a symmetric 2ℓ-linear form for a fixed $u\in H$ and $\tilde{g}_i(x,y;u,v)$ is a symmetric bilinear form for fixed $x,y\in H$. $\square$

We shall prove the theorem by induction on n. It is obvious that $\|f\|_1=\|f\|_2$ if $n=1$. Let $\ell\in\mathbb{N}$, and suppose that the theorem is valid for $n\le\ell$. Let $n=\ell+1$ and $\varepsilon>0$. Then there are $u_j\in H$ ($1\le j\le n$) satisfying $\|u_j\|=1$ and

$$|f(u_1,u_2,\cdots,u_n)|\ge\|f\|_2-\varepsilon.$$

Let $X=\{\sum_{j=1}^{n}a_ju_j;\ a_j\in\mathbb{K}\ (\ 1\le j\le n)\}$. We may assume that

$$\sup\{|f(x_1,\cdots,x_n)|;\ x_j\in X\text{ and }\|x_j\|=1\ (\ 1\le j\le n)\}=1.$$

Let us show that

$$\sup\{|f(x,\cdots,x)|;\ x\in X\text{ and }\|x\|=1\}\tag{2}$$

$$\geq \sup\{|f(x_1,\cdots,x_n)|;\ x_j \in X \text{ and } \|x_j\| = 1\ (\ 1 \leq j \leq n)\} = 1$$
$$(\geq \|f\|_2 - \varepsilon),$$

which proves the theorem. Let $1 \leq k < n$. By the induction assumption we have

$$\sup\{|f(x_1,\cdots,x_k,x_{k+1},\cdots,x_n)|;\ x_j \in X \text{ and } \|x_j\| = 1\ (\ 1 \leq j \leq k)\}$$
$$= \sup\{|f(x,\cdots,x,x_{k+1},\cdots,x_n)|;\ x \in X \text{ and } \|x\| = 1\}$$

for any $x_{k+1},\cdots,x_n \in X$. Moreover, we have

$$\begin{aligned} 1 = \sup\{|f(x,\cdots,x,x_{k+1},\cdots,x_n)|;\ & x,x_{k+1},\cdots,x_n \in X \\ & \text{and } \|x\| = \|x_{k+1}\| = \cdots = \|x_n\| = 1\} \\ = \sup\{|f_k(x,y)|;\ x,y \in X \text{ and } \|x\| = \|y\| = 1\}, & \end{aligned}$$

where

$$f_i(x,y) = f(x_1,\cdots,x_n)|_{x_j = x\ (1\leq j \leq i),\ x_h = y\ (i+1 \leq h \leq n)}.$$

Put

$$\begin{aligned} V = \{(x,y) \in X \times X;\ & \|x\| = \|y\| = 1 \\ & \text{and } |f_k(x,y)| = 1 \text{ for some } k \text{ with } 1 \leq k \leq n-1\}. \end{aligned}$$

Since X is a finite dimensional subspace, we have $V \neq \emptyset$. Define

$$\gamma = \max\{|(x,y)_H|;\ (x,y) \in V\},$$

and choose $(x,y) \in V$ and $k \in \mathbb{N}$ so that $1 \leq k \leq n-1$, $(x,y)_H = \gamma$ and $|f_k(x,y)| = 1$. Since $(v,u) \in V$ if $(u,v) \in V$, we may assume that $k \leq n/2$. Applying (1) to a symmetric $2k$-linear form $f(x_1,\cdots,x_{2k},y,\cdots,y)$, we have

$$\begin{aligned} 2^{2k} = 2^{2k}|f_k(x,y)| &\leq \sum_{j=0}^{k} \binom{k}{j} |\tilde{f}_{2j,k}(x+y,x-y;y)| \\ &= |f_{2k}(x+y,y)| + \sum_{j=0}^{k-1} \binom{k}{j} |\tilde{f}_{2j,k}(x+y,x-y;y)| \\ &\leq |f_{2k}(x+y,y)| + \Big(\sum_{j=0}^{k} \binom{k}{j} \|x+y\|^{2j} \|x-y\|^{2(k-j)} - \|x+y\|^{2k} \Big) \\ &= |f_{2k}(x+y,y)| - \|x+y\|^{2k} + \sum_{j=0}^{k} \binom{k}{j} (2+2\gamma)^j (2-2\gamma)^{k-j} \\ &= |f_{2k}(x+y,y)| - \|x+y\|^{2k} + 2^{2k}, \end{aligned}$$

where

$$\tilde{f}_{i,k}(u,v;w) = f(x_1,\cdots,x_{2k},w,\cdots,w)|_{x_j=u\ (1\leq j\leq i),\ x_h=v\ (i+1\leq h\leq 2k)}.$$

This gives $|f_{2k}(x+y,y)| \geq \|x+y\|^{2k}$, which implies that $|f_{2k}(x+y,y)| = \|x+y\|^{2k}$. Putting $w = (x+y)/\|x+y\|$, we have $|f_{2k}(w,y)| = 1$ and $\|w\| = 1$. If $2k = n$, then $|f_{2k}(w,y)| = |f(w,\cdots,w)| = 1$. In this case (2) holds. Now assume that $2k < n$. Then we have $(w,y) \in V$ and

$$(w,y)_H = \sqrt{(\gamma+1)/2} \leq \gamma. \tag{3}$$

(3) and the definition of γ yield $\gamma = 1$ and, therefore, $x = y$, which implies that $|f(x,\cdots x)| = 1$ and (2) holds.