Is there $g(t) \in C^\infty(\mathbb{R})$ satisfying $f(t) = g(t)^2$ when $f(t) \in C^\infty(\mathbb{R})$ and $f(t) \geq 0$?

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May 19, 2006

We shall prove the following

Theorem 1. *Let* $f(t) \in C^\infty(\mathbb{R})$ *satisfy* $f(t) \geq 0$ ($t \in \mathbb{R}$), *and assume that* $Z \equiv \{t \in \mathbb{R};\ f(t) = 0\}$ *has no accumulating point. Moreover, we assume the following condition:*

(A) *If* $t_0 \in Z$ *is a zero of* f *of infinite order, then there are* $N \in \mathbb{N}$ *and* $\eta > 0$ *such that for any* $\varepsilon > 0$ *there is* $C_\varepsilon > 0$ *satisfying*

$$\int_0^1 (1-\theta)^{N-1} f(t_0 + \theta t)\, d\theta \leq C_\varepsilon f(t_0 + t)^{1-\varepsilon}$$

for $t \in [t_0 - \eta, t_0 + \eta]$.

Then there is $g(t) \in C^\infty(\mathbb{R})$ *satisfying* $f(t) = g(t)^2$ ($t \in \mathbb{R}$).

Remark. If for every $t_0 \in Z$ there is $\eta > 0$ such that $f(t) \searrow$ on $[t_0 - \eta, t_0]$ and $f(t) \nearrow$ on $[t_0, t_0 + \eta]$, then the condition (A) is satisfied. In particular, if the set $\{t \in \mathbb{R};\ f'(t) = 0\}$ has no accumulating point, then the condition (A) is satisfied.

To prove the theorem we need the following two lemmas.

Lemma 2. *Assume that* $f(t) \in \mathscr{B}(\mathbb{R})$, *i.e.,* $f(t)$ *and its derivatives are all bounded, and that for any* $\varepsilon > 0$ *there is* $C_\varepsilon > 0$ *satisfying*

$$|f(s)| \leq C_\varepsilon |f(t)|^{1-\varepsilon} \quad \textit{if } -\infty < s < t < \infty. \tag{1}$$

Then, for any $k \in \mathbb{Z}_+$ ($:= \mathbb{N} \cup \{0\}$) *and* $\varepsilon > 0$ *there is* $C_{k,\varepsilon} > 0$ *such that*

$$|f^{(k)}(t)| \leq C_{k,\varepsilon} |f(t)|^{1-\varepsilon} \quad (t \in \mathbb{R}).$$

Proof. For fixed $N \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$f(t+s) = f(t) + sf'(t) + \cdots + \frac{s^{N-1}}{(N-1)!} f^{(N-1)}(t) + \frac{s^N}{N!} f^{(N)}(t+\theta s),$$

where $0 < \theta \equiv \theta_N(t,s) < 1$. Now assume that $f(t) \neq 0$. We have

$$\begin{aligned} h_j(t) &:= f(t) - f(t - j|f(t)|^{1/N}) + \frac{(-1)^N j^N}{N!} |f(t)| f^{(N)}(t - j\theta_j |f(t)|^{1/N}) \\ &= j|f(t)|^{1/N} f'(t) - \frac{j^2}{2} |f(t)|^{2/N} f''(t) + \\ &\quad \cdots + (-1)^N \frac{j^{N-1}}{(N-1)!} |f(t)|^{(N-1)/N} f^{(N-1)}(t) \quad (\, j = 1, 2, \cdots, N-1), \end{aligned}$$

where $(0<)\theta_j = \theta_N(t, -j|f(t)|^{1/N})(<1)$. Note that

$$|h_j(t)| \leq |f(t)|(1 + j^N C_N(f)/N!) + C_\delta |f(t)|^{1-\delta} \leq C_{N,\delta} |f(t)|^{1-\delta} \quad \text{for } \delta > 0,$$

where $C_N(f) = \sup_{s \in \mathbb{R}} |f^{(N)}(s)|$, C_δ is the constant in (1) and the $C_{N,\delta}$ are positive constants. Put

$$A = ((-1)^{k+1} j^k)_{\substack{j \downarrow 1,2,\cdots,N-1 \\ k \rightarrow}} .$$

Since $\det A \neq 0$, there is the inverse $A^{-1} = (B_{N,j,k})_{\substack{j \downarrow 1,2,\cdots,N-1 \\ k \rightarrow}}$. Therefore, we have, with some positive constants $C_{N,j,\delta}$,

$$\begin{aligned} &|f(t)|^{j/N} f^{(j)}(t)/j! = \sum_{k=1}^{N-1} B_{N,j,k} h_k(t), \\ &|f^{(j)}(t)| \leq C_{N,j,\delta} |f(t)|^{1-\delta-j/N} \quad \text{for } \delta > 0 \end{aligned}$$

$(\, j \leq j \leq N-1)$. □

Lemma 3. *Assume that* $f(t) \in \mathscr{B}(\mathbb{R})$*,* $f(t) = 0$ *(* $t < 0$*) and* $f(t) > 0$ *(* $t > 0$*). Moreover, we assume that there is* $N \in \mathbb{N}$ *such that for any* $\varepsilon > 0$ *there is* $C_\varepsilon > 0$ *satisfying*

$$\int_0^1 (1-\theta)^{N-1} f(\theta t)\, d\theta \leq C_\varepsilon f(t)^{1-\varepsilon} \quad (\, 0 < t \leq 1). \tag{2}$$

Then, for any $k \in \mathbb{Z}_+$ *and* $\varepsilon > 0$ *there is* $C_{k,\varepsilon} > 0$ *such that*

$$|f^{(k)}(t)| \leq C_{k,\varepsilon} f(t)^{1-\varepsilon} \quad (\, t \leq 1). \tag{3}$$

Moreover, for $n \in \mathbb{N}$ $f(t)^{1/n}$ *belongs to* $C^\infty(\mathbb{R})$*.*

Remark. If for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $f(s) \le C_\varepsilon f(t)^{1-\varepsilon}$ ($0 < s < t \le 1$), then for any $N \in \mathbb{N}$ (2) is valid.

Proof. Put

$$F(t) = \int_0^1 (1-\theta)^{N-1} f(\theta t)\, d\theta \cdot t^N/(N-1)!.$$

Then we have $F^{(N)}(t) = f(t)$. Indeed, we have

$$\begin{aligned} F'(t) &= \int_0^1 \theta(1-\theta)^{N-1} f'(\theta t)\, d\theta \cdot t^N/(N-1)! \\ &\quad + \int_0^1 (1-\theta)^{N-1} f(\theta t)\, d\theta \cdot N t^{N-1}/(N-1)! \\ &= \Big\{ [\theta(1-\theta)^{N-1} f(\theta t)]_{\theta=0}^{\theta=1} - \int_0^1 ((1-\theta)^{N-1} - (N-1)\theta(1-\theta)^{N-2}) \\ &\quad \times f(\theta t)\, d\theta \Big\} t^{N-1}/(N-1)! + \int_0^1 (1-\theta)^{N-1} f(\theta t)\, d\theta \cdot N t^{N-1}/(N-1)! \\ &= \begin{cases} \displaystyle\int_0^1 (1-\theta)^{N-2} f(\theta t)\, d\theta \cdot t^{N-1}/(N-2)! \quad (N \ge 2), \\ f(t) \quad (N = 1), \end{cases} \end{aligned}$$

which proves the assertion. Modifying $F(t)$ for $t \ge 1$, we may assume that $F(t) \in \mathscr{B}(\mathbb{R})$ and $F(t) \nearrow$. Since $0 \le F(s) \le F(t)$ ($s < t$), it follows from Lemma 2 and (2) that for any $k \in \mathbb{Z}_+$ and $\varepsilon > 0$ there are positive constants $C_{k,\varepsilon}$ and $C'_{k,\varepsilon}$ such that

$$|f^{(k)}(t)| \le C'_{k,\varepsilon} F(t)^{1-\varepsilon/2} \le C_{k,\varepsilon} f(t)^{1-\varepsilon} \quad (t \le 1),$$

which proves the first part of the lemma. Put $g_n(t) = f(t)^{1/n}$ for $n \in \mathbb{N}$. Let $\delta > 0$. Then we can show by induction that there are $C_k > 0$ ($k \in \mathbb{Z}_+$) satisfying

$$|g_n^{(k)}(t)| \le C_k f(t)^{1/n - k\delta} \quad (k \in \mathbb{Z}_+,\ 0 < t \le 1). \tag{4}$$

Indeed, (4) with $k = 0$ is valid. Suppose that (4) holds for $k \le \ell$. From the identity $n g'_n(t) g_n(t)^{n-1} = f'(t)$, we have

$$n g_n^{(\ell+1)}(t) g_n(t)^{n-1} = f^{(\ell+1)}(t) - n \sum_{|\alpha|=\ell,\, \alpha_1<\ell} \frac{\ell!}{\alpha!} g_n^{(\alpha_1+1)}(t) g_n^{(\alpha_2)}(t) \cdots g_n^{(\alpha_n)}(t),$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{Z}_+)^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. This, together with (3) and (4) for $k \le \ell$, gives

$$|g_n^{(\ell+1)}(t)| \le n^{-1} f(t)^{-1+1/n} \{ C_{\ell+1,\delta(\ell+1)} f(t)^{1-(\ell+1)\delta}$$

$$
\begin{aligned}
&+n\sum_{|\alpha|=\ell,\,\alpha_1<\ell}\frac{\ell!}{\alpha!}C_{\alpha_1+1}C_{\alpha_2}\cdots C_{\alpha_n}f(t)^{1-(\ell+1)\delta}\}\\
&\le C_{\ell+1}f(t)^{1/n-(\ell+1)\delta}\quad (\,0<t\le 1),
\end{aligned}
$$

which proves (4). Note that $\lim_{t\to+0}g_n^{(k)}(t)=0$ ($k\in\mathbb{Z}_+$). Applying the mean value theorem, we can prove inductively $g_n(t)\in C^\infty(\mathbb{R})$. $\square$

Now we can prove Theorem 1. We can assume without loss of generality that

$$
Z=\{t_0,t_{\pm1},t_{\pm2},\cdots\}\quad\text{and}\quad \cdots<t_{-2}<t_{-1}<0\le t_0<t_1<t_2<\cdots.
$$

If t_0 is a zero of f of infinite order, then by Lemma 3 we can choose $g(t)\ge 0$ near t_0 so that $g(t)$ is in C^∞ and $g(t)^2=f(t)$. In the case t_0 is a zero of order $4m$ ($m=1,2,\cdots$), we can also construct $g(t)\ge 0$ near t_0. If t_0 is a zero of order $4m-2$ ($m=1,2,\cdots$), then we can construct $g(t)$ near t_0 so that $g(t)<0$ for $t\in(t_{-1},t_0)$ and $g(t)>0$ for $t\in(t_0,t_1)$. Next we extend $g(t)$ in a neighborhood of $t=t_1$, choosing its signature appropriately. And then we extend $g(t)$ in a neighborhoods of $t=t_{-1}$, $t=t_2$, $t=t_{-2}$, $\cdots$ in turn. Finally we obtain $g(t)\in C^\infty(\mathbb{R})$ satisfying $g(t)^2=f(t)$.

Example 1. Let $a\ge 2$, and put

$$
f(t)=\begin{cases} e^{-1/t}(1-\sin(1/t))+e^{-a/t} & (\,t>0),\\ 0 & (\,t\le 0).\end{cases}
$$

Then $f(t)>0$ ($t>0$) and $\sqrt{f(t)}\notin C^2(\mathbb{R})$. Indeed, putting $t_n=(2n\pi+\pi/2)^{-1}$ ($n\in\mathbb{N}$), we have

$$
\begin{aligned}
&f(t_n)=e^{-a/t_n},\quad f'(t_n)=at_n^{-2}e^{-a/t_n},\\
&f''(t_n)=t_n^{-4}e^{-1/t_n}+a^2t_n^{-4}e^{-a/t_n}-2at_n^{-3}e^{-a/t_n}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\frac{d^2}{dt^2}\sqrt{f(t)}|_{t=t_n}=f''(t_n)f(t_n)^{-1/2}/2-f'(t_n)^2f(t_n)^{-3/2}/4\\
&=t_n^{-4}e^{(a/2-1)/t_n}+o(1)\quad\text{as } n\to\infty.
\end{aligned}
$$

This gives $\lim_{n\to\infty}(d^2/dt^2)\sqrt{f(t)}|_{t=t_n}=\infty$. One can also find a similar example in the following paper:

G. Glaeser, Racine carrée d’une fonction différentiable, Ann. Inst. Fourier, Grenoble **13** (1963), 203–210.