An alternative proof of Ivrii-Petkov's necessary condition for C^∞ well-posedness of the Cauchy problem

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May 17, 2006

Let $P(x,\xi)$ be a polynomial of $\xi = (\xi_1, \cdots, \xi_n)$ whose coefficients are C^∞ functions of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We write

$$P(x,\xi) = \sum_{j=0}^{m} P_j(x,\xi),$$

where $m = \deg_\xi P(x,\xi)$ and $P_j(x,\xi)$ is a homogeneous polynomial of degree j. Let us consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(x,D)u(x) = f(x) \quad \text{in } \mathbb{R}^n, \\ \operatorname{supp} u \subset \{x \in \mathbb{R}^n;\ x_1 \geq 0\}, \end{cases}$$

where $D = (D_1, \cdots, D_n) = -i(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $f \in C^\infty(\mathbb{R}^n)$ satisfies supp $f \subset \{x \in \mathbb{R}^n;\ x_1 \geq 0\}$. We say that the Cauchy problem (CP) is C^∞ well-posed if the following two conditions are satisfied:

(E) For any $f \in C^\infty(\mathbb{R}^n)$ with supp $f \subset \{x \in \mathbb{R}^n;\ x_1 \geq 0\}$ there is $u \in C^\infty(\mathbb{R}^n)$ satisfying (CP).

(U) If $t > 0$, $u \in C^\infty(\mathbb{R}^n)$, supp $u \subset \{x \in \mathbb{R}^n; x_1 \geq 0\}$ and supp $P(x,D)u \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$, then supp $u \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$.

We assume that $P_m(x,\vartheta) \neq 0$, where $\vartheta = (1,0,\cdots,0) \in \mathbb{R}^n$. Then C^∞ well-posedness implies that $P_m(x,\xi)$ is hyperbolic with respect to ϑ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$, *i.e.*, $P_m(x,\xi - i\vartheta) \neq 0$ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$ and $\xi \in \mathbb{R}^n$ (see [Mi]). Therefore, we assume that $P_m(x,\xi)$ is hyperbolic with respect to ϑ for each $x \in \mathbb{R}^n$ with $x_1 \geq 0$.

Ivrii and Petkov gave a necessary condition for C^∞ well-posedness in [IP], and Ivrii improved the result and gave the following theorem in [I] (see, also Mandai [Ma]).

Theorem 1. *Assume that the Cauchy problem* (CP) *is C^∞ well-posed. Let $x^0 \in \mathbb{R}^n$ satisfy $x_1^0 \ge 0$, and assume that there are $r \in \mathbb{Z}_+$ ($:= \mathbb{N} \cup \{0\}$) and $q_j \in \mathbb{Q}$ ($1 \le j \le n$) such that $q_j > 0$ ($1 \le j \le n$), $1 + q_1 > q_j$ ($2 \le j \le n$) and*

$$
\begin{aligned}
&P_m^{(re_1)}(x^0, e_n) \ne 0, \\
&P_{m(\beta)}^{(\alpha)}(x^0, e_n) = 0 \quad \textit{if } (1+q_1)|\alpha| + \langle q, \beta - \alpha \rangle < r,
\end{aligned}
$$

where e_j denotes the n-tuple vector whose k-th component is equal to $\delta_{j,k}$ ($1 \le k \le n$), $P_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta P(x,\xi)$, $q = (q_1, \cdots, q_n)$ and $\langle q, \beta - \alpha \rangle = \sum_{j=1}^n q_j(\beta_j - \alpha_j)$ for $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in (\mathbb{Z}_+)^n$. Then

$$
P_{m-s(\beta)}^{(\alpha)}(x^0, e_n) = 0 \quad \textit{if } (1+q_1)(s+|\alpha|) + \langle q, \beta - \alpha \rangle < r.
$$

Remark. We note that $P_{m-s(\beta)}^{(\alpha')}(x^0, e_n) \ne 0$ if $\alpha = (\alpha', \alpha_n) \in (\mathbb{Z}_+)^n$, and $P_{m-s(\beta)}^{(\alpha)}$ $(x^0, e_n) \ne 0$, where $P_{(\beta)}^{(\alpha')}(x,\xi) = P_{(\beta)}^{((\alpha',0))}(x,\xi)$.

We shall prove the above theorem, repeating the same argument as in the first part of the proof in [Ma] and, then, applying the idea used in [W].

Now we assume that the hypotheses of Theorem 1 are fulfilled. From Banach's closed graph theorem or the Baire category theorem we have the following lemma (see, *e.g.*, [IP]).

Lemma 2. *Let K be a compact subset of $\{x \in \mathbb{R}^n;\ x_1 \ge 0\}$. Then there are $\ell \equiv \ell_K \in \mathbb{Z}_+$ and $C \equiv C_K > 0$ such taht*

$$
|u(x^1)| \le C \sup_{|\beta| \le \ell} \sup_{x_1 \le x_1^1} |D^\beta (P(x,D)u(x))|
$$

if $x^1 \in K$, $u \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\, u \subset K$.

Let $\delta > 0$. We make an asymptotic change of variables

$$
y = \rho^{\delta q}(x - x^0) = (\rho^{\delta q_1}(x_1 - x_1^0), \cdots, \rho^{\delta q_n}(x_n - x_n^0)) \quad (\ \rho \gg 1).
$$

Put $P_\rho(y,\eta) = P(x^0 + \rho^{-\delta q} y, \rho^{\delta q}\eta)$. Then we have

$$
\begin{aligned}
P_\rho(y,\eta) = &\sum_{\substack{0 \le s \le m,\, \alpha' \in (\mathbb{Z}_+)^{n-1} \\ \beta \in (\mathbb{Z}_+)^n,\, \mu(s,\alpha',\beta) > -N}} \rho^{\mu(s,\alpha',\beta)} \frac{y^\beta}{\alpha'!\beta!} P_{m-s(\beta)}^{(\alpha')}(x^0, e_n) \eta^{\alpha'} \eta_n^{m-s-|\alpha'|} \\
&+ \rho^{-N} R_N(y,\eta;\rho) \\
=: &\ Q_N(y,\eta;\rho) + \rho^{-N} R_N(y,\eta;\rho),
\end{aligned}
$$

where $N \in \mathbb{N}$, $\eta^{\alpha'} = \eta_1^{\alpha_1} \cdots \eta_{n-1}^{\alpha_{n-1}}$ for $\alpha' = (\alpha_1, \cdots, \alpha_{n-1})$, $q' = (q_1, \cdots, q_{n-1})$ and

$$\mu(s, \alpha', \beta) = \delta \langle q', \alpha' - \beta' \rangle + \delta q_n (m - s - |\alpha'| - \beta_n).$$

Write

$$R_N(y, \eta; \rho) = \sum_{|\alpha| \le m} R_{N,\alpha}(y; \rho) \eta^\alpha.$$

Then for any $N \in \mathbb{N}$ and $W \Subset \mathbb{R}^n$ there are $C_{N,W,\beta} > 0$ ($\beta \in (\mathbb{Z}_+)^n$) such that

$$|R_{N,\alpha(\beta)}(y; \rho)| \le C_{N,W,\beta} \quad \text{for } y \in W \text{ and } \rho \ge 1.$$

Put

$$\begin{aligned} \mathfrak{M} = \{(s, \alpha', \beta);\ & 1 \le s \le m,\ (1 + q_1)(s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle < r \\ & \text{and } P^{(\alpha')}_{m-s(\beta)}(x^0, e_n) \ne 0\} \end{aligned}$$

By assumption $\mathfrak{M}$ is a finite set. Note that $(s, \alpha', \beta) \in \mathfrak{M}$ if $\alpha = (\alpha', \alpha_n)$, $1 \le s \le m$, $(1 + q_1)(s + |\alpha|) + \langle q, \beta - \alpha \rangle < r$ and $P^{(\alpha)}_{m-s(\beta)}(x^0, e_n) \ne 0$. So, in order to prove Theorem 1 it suffices to show that $\mathfrak{M} = \emptyset$. Now suppose that $\mathfrak{M} \ne \emptyset$. Define

$$\begin{aligned} \varepsilon_0 = \max\{\varepsilon;\ & \varepsilon > 0 \text{ and } (1 + q_1)(\varepsilon s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r \\ & \text{for some } (s, \alpha', \beta) \in \mathfrak{M}\} \quad (> 1), \\ \mathfrak{M}_0 = \{(s, \alpha', \beta) \in \mathfrak{M};\ & (1 + q_1)(\varepsilon_0 s + |\alpha'|) + \langle q, \beta \rangle - \langle q', \alpha' \rangle = r\}. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{\mu}(s, \alpha', \beta) &\equiv \mu(s, \alpha', \beta) + m - s - |\alpha'| + \sigma |\alpha'| \\ &= \mu_0 + (r - |\alpha'|)(1 - \sigma - \delta(1 + q_1 - q_n)) + s\{\delta((1 + q_1)\varepsilon_0 - q_n) - 1\} \\ &\qquad \text{for } (s, \alpha', \beta) \in \mathfrak{M}_0, \end{aligned}$$

where $\mu_0 = (\delta q_1 + \sigma) r + (1 + \delta q_n)(m - r)$. We choose $\delta > 0$ and $\sigma > 0$ so that

$$\delta(1 + q_1 - q_n) = 1 - \sigma, \quad \delta((1 + q_1)\varepsilon_0 - q_n) = 1,$$

i.e.,

$$\delta = ((1 + q_1)\varepsilon_0 - q_n)^{-1}, \quad \sigma = (1 + q_1)(\varepsilon_0 - 1)((1 + q_1)\varepsilon_0 - q_n)^{-1}.$$

Note that $0 < \sigma < 1$. By this choice we have the following:
(i) $\tilde{\mu}(s, \alpha', \beta) = \mu_0$ for $(s, \alpha', \beta) \in \mathfrak{M}_0$.
(ii) $\tilde{\mu}(s, \alpha', \beta) < \mu_0$ for $(s, \alpha', \beta) \in \mathfrak{M} \setminus \mathfrak{M}_0$.

(iii) $\tilde{\mu}(s,\alpha',\beta) < \mu_0$

if $1 \le s \le m$ and $(1+q_1)(s+|\alpha'|)+\langle q,\beta\rangle - \langle q',\alpha'\rangle \ge r$.

(iv) $\tilde{\mu}(0,\alpha',\beta) = \mu_0$ if $(1+q_1)|\alpha'|+\langle q,\beta\rangle - \langle q',\alpha'\rangle = r$.

(v) $\tilde{\mu}(0,\alpha',\beta) < \mu_0$ if $(1+q_1)|\alpha'|+\langle q,\beta\rangle - \langle q',\alpha'\rangle > r$.

Put

$$\mathfrak{M}_1 = \mathfrak{M}_0 \cup \{(0,\alpha',\beta);\ (1+q_1)|\alpha'|+\langle q,\beta\rangle - \langle q',\alpha'\rangle = r \text{ and } P^{(\alpha')}_{m(\beta)}(x^0,e_n) \neq 0\}.$$

Then there is $\delta_0 > 0$ such that for $N \gg 1$ and $\gamma \in \mathbb{R}^n \setminus \{0\}$

$$(1) \qquad Q_N(y,\gamma\rho e_n + \rho^\sigma \eta;\rho) = \rho^{\mu_0}\{\gamma^{m-r}\Phi(y,\eta';\gamma) + \rho^{-\delta_0} r_N(y,\eta;\rho,\gamma)\},$$

where

$$\Phi(y,\eta';\gamma) = \sum_{(s,\alpha',\beta)\in\mathfrak{M}_1} \frac{\gamma^{r-|\alpha'|-s}y^\beta}{\alpha'!\beta!} P^{(\alpha')}_{m-s(\beta)}(x^0,e_n)\eta^{\alpha'}.$$

Here $r_N(y,\eta;\rho,\gamma)$ is a polynomial of (y,η,γ) and its coefficients are bounded for $\rho \ge 1$.

Lemma 3. (i) *If* $(s,\alpha',\beta) \in \mathfrak{M}_1$, *then* $(s,\alpha',\beta) = (0,re'_1,0)$ *or* $\alpha_1 + s < r$, *where* $e'_1 = (1,0,\cdots,0) \in (\mathbb{Z}_+)^{n-1}$. (ii) *There are* $\hat{y} \in \mathbb{R}^n$, $\hat{\eta}' \in \mathbb{C}^n \times (\mathbb{R}^{n-2} \setminus \{0\})$ *and* $\hat{\gamma} \in \mathbb{R} \setminus \{0\}$ *such that* $\hat{y}_1 > 0$ *if* $x^0_1 = 0$, Im $\hat{\eta}_1 < 0$ *and* $\Phi(\hat{y},\hat{\eta}';\hat{\gamma}) = 0$.

Proof. (i) Let $(0,\alpha',\beta) \in \mathfrak{M}_1$. Then we have

$$\alpha_1 + \langle q,\beta\rangle + \sum_{j=2}^{n-1}(1+q_1-q_j)\alpha_j = r$$

Since $1+q_1 > q_j$ ($2 \le j \le n-1$), we have $\alpha_1 < r$ if $\sum_{j=2}^{n-1}\alpha_j + |\beta| \neq 0$. By assumption we have $(0,re'_1,0) \in \mathfrak{M}_1$. Moreover, if $(s,\alpha',\beta) \in \mathfrak{M}_0$, we have

$$\alpha_1 + s \le (1+q_1)(s+|\alpha'|) + \langle q,\beta\rangle - \langle q',\alpha'\rangle < r.$$

(ii) Put

$$\theta = \max\{(\varepsilon_0-1)(1+q_1)s/(r-\alpha_1);\ (s,\alpha',\beta)\in\mathfrak{M}_0\}\ (>0),$$
$$\mathfrak{M}' = \{(s,\alpha',\beta)\in\mathfrak{M}_1;\ \theta(r-\alpha_1) = (\varepsilon_0-1)(1+q_1)s\}.$$

Note that

$$(2) \qquad (0,\alpha',\beta) \in \mathfrak{M}' \text{ if and only if } \alpha' = re'_1 \text{ and } \beta = 0.$$

For $\omega \gg 1$ we have

$$\Phi(\omega^{-q}y, \omega^{\tilde{q}}\eta'; \omega^{1+q_1}\gamma)$$

$$= \sum_{(s,\alpha',\beta)\in\mathfrak{M}_1} \frac{\omega^{\nu(s,\alpha',\beta)}}{\alpha'!\beta!}\gamma^{r-|\alpha'|-s}y^\beta P^{(\alpha')}_{m-s(\beta)}(x^0,e_n)\eta^{\alpha'},$$

where $\tilde{q} = (q_1+\theta, q_2, \cdots, q_n) \in \mathbb{R}^{n-1}$ and $\nu(s,\alpha',\beta) = (1+q_1)(r-|\alpha'|-s) - \langle q,\beta\rangle + \langle q',\alpha'\rangle + \theta\alpha_1$. Since

$$\begin{aligned}\nu(s,\alpha',\beta) = &-\{(1+q_1)(\varepsilon_0 s+|\alpha'|)+\langle q,\beta\rangle - \langle q',\alpha'\rangle\}\\ &+(\varepsilon_0-1)(1+q_1)s+(1+q_1)r+\theta\alpha_1,\end{aligned}$$

we have

$$\nu(s,\alpha',\beta)\begin{cases} = (q_1+\theta)r & \text{if } (s,\alpha',\beta)\in\mathfrak{M}',\\ < (q_1+\theta)r & \text{if } (s,\alpha',\beta)\in\mathfrak{M}_1\setminus\mathfrak{M}'.\end{cases}$$

Putting

$$\Phi_1(y,\eta';\gamma) = \sum_{(s,\alpha',\beta)\in\mathfrak{M}'} \frac{\gamma^{r-|\alpha'|-s}y^\beta}{\alpha'!\beta!}P^{(\alpha')}_{m-s(\beta)}(x^0,e_n)\eta^{\alpha'},$$

we have

$$\begin{aligned}&\Phi(\omega^{-q}y,\omega^{\tilde{q}}\eta';\omega^{1+q_1}\gamma) = \omega^{(q_1+\theta)r}(\Phi_1(y,\eta';\gamma)+o(1)) \quad \text{as } \omega\to\infty,\\ &\Phi_1(y,\eta_1,\gamma\eta''';\gamma) = \sum_{(s,\alpha',\beta)\in\mathfrak{M}'}\frac{\gamma^{r-\alpha_1-s}y^\beta}{\alpha'!\beta!}P^{(\alpha')}_{m-s(\beta)}(x^0,e_n)\eta^{\alpha'''}\eta_1^{\alpha_1},\end{aligned}$$

where $\eta''' = (\eta_2,\cdots,\eta_{n-1})$, $\alpha' = (\alpha_1,\alpha''') \in (\mathbb{Z}_+)^{n-1}$ and $\eta^{\alpha'''} = \eta^{(0,\alpha''',0)}$. It follows from the assertion (i) and (2) that

$$\begin{aligned}&\Phi_1(y,\eta_1,\gamma\eta''';\gamma) = P^{(re_1)}_m(x^0,e_n)\eta_1^r/r! + \sum_{j=0}^{r-1}\sum_{s=1}^{m}\gamma^{r-j-s}A_{j,s}(y,\eta''')\eta_1^j,\\ &r-2\ge r-s-1\ge j \quad \text{if } A_{j,s}(y,\eta''')\not\equiv 0.\end{aligned}$$

Choose $\tilde{y}\in\mathbb{R}^n$, $\tilde{\eta}'''\in\mathbb{R}^{n-2}\setminus\{0\}$ and $j^*,s^*\in\mathbb{Z}_+$ so that $\tilde{y}_1>0$, $0\le j^*\le r-2$, $1\le s^*\le m$ and $A_{j^*,s^*}(\tilde{y},\tilde{\eta}''')\ne 0$. For example, if $\gamma\in\mathbb{R}\setminus\{0\}$ and $|\gamma|$ is sufficiently small, then the equation $\Phi_1(\tilde{y},\eta_1,\gamma\tilde{\eta}''';\gamma)=0$ in η_1 has a root with negative imaginary part for $\gamma>0$ or $\gamma<0$. Indeed, putting $\kappa = \min\{(r-j-s)/(r-j);\ A_{j,s}(\tilde{y},\tilde{\eta}''')\ne 0\}$, we have $0<\kappa<1$. Moreover, the roots of $\Phi_1(\tilde{y},\eta_1,\gamma\tilde{\eta}''';\gamma)=0$ in η_1 can be expanded in Puiseux series with respect to γ and $\Phi_1(\tilde{y},\eta_1,\gamma\tilde{\eta}''';\gamma)=0$ in η_1 has a root $\eta_1 = a\gamma^\kappa(1+o(1))$ as $\gamma\to 0$ with $a\ne 0$. In particular, there are $\tilde{\gamma}\in\mathbb{R}\setminus\{0\}$ and $\tilde{\eta}_1\in\mathbb{C}$ such that $\operatorname{Im}\tilde{\eta}_1<0$ and $\Phi_1(\tilde{y},\tilde{\eta}_1,\tilde{\gamma}\tilde{\eta}''';\tilde{\gamma})=0$. This proves the assertion (ii) with $\omega\gg 1$, $\hat{y}=\omega^{-q}\tilde{y}$, $\hat{\eta}'=\omega^{\tilde{q}}(\tilde{\eta}_1,\tilde{\gamma}\tilde{\eta}''')$ and $\hat{\gamma}=\omega^{1+q_1}\tilde{\gamma}$. □

From Lemma 3 there are $\eta^{0'''}\in\mathbb{R}^{n-2}\setminus\{0\}$, an open neighborhood U of $\eta^{0'''}$, $p\in\mathbb{N}$ and real analytic functions $\tau(\eta''')$ and $\widetilde{\Phi}(\eta_1,\eta''')$ defined for $\eta'''\in U$ such that $1 \le p \le r$, Im $\tau(\eta''') < 0$, $\widetilde{\Phi}(\eta')$ is a polynomial of η_1, $\widetilde{\Phi}(\tau(\eta'''), \eta''') \neq 0$ and

$$\Phi(\hat{y}, \eta_1, \eta'''; \hat{\gamma}) = (\eta_1 - \tau(\eta'''))^p \widetilde{\Phi}(\eta').$$

Let $\varphi(x)$ be a solution of

$$\frac{\partial \varphi}{\partial y_1} = \tau(\nabla_{y'''} \varphi(y)), \quad \varphi(\hat{y}_1, y'') = (y''' - \hat{y}''') \cdot \eta^{0'''} + i|y'' - \hat{y}''|^2,$$

in an open neighborhood V of $\hat{y}$, where $y = (y', y_n) = (y_1, y''', y_n) = (y_1, y'')$. We may assume that $x_1^0 + \rho^{-\delta q_1} y_1 > 0$ if $y \in V$ and $\rho \ge 1$. Up to this point the proof is the same as in [Ma]. It follows from §3 of Chapter VI of [T] that

$$\begin{aligned}
&\Phi(\hat{y}, \rho^{-\sigma} D'; \hat{\gamma})(\exp[i\rho^\sigma \varphi(y)] u(y)) \\
&= \exp[i\rho^\sigma \varphi(y)] \sum_{|\alpha'| \ge p} \Phi^{(\alpha')}(\hat{y}, \nabla_{y'} \varphi(y); \hat{\gamma}) \mathfrak{N}_{\alpha'}(y, D'; \rho) u(y), \\
&\mathfrak{N}_{\alpha'}(y, D'; \rho) u(y) = \rho^{-\sigma|\alpha'|} [D_{w'}^{\alpha'}(\exp[i\rho^\sigma \Psi(y, w')] u(w', y_n))]_{w'=y'} / \alpha'!,
\end{aligned}$$

where $D' = (D_1, \cdots, D_{n-1})$, $D_{w'}^{\alpha'} = D_{w_1}^{\alpha_1} \cdots D_{w_{n-1}}^{\alpha_{n-1}}$ and $\Psi(y, w') = \varphi(w', y_n) - \varphi(y) - (w' - y') \cdot (\nabla_{y'} \varphi)(y)$. It is easy to see that

$$\begin{aligned}
&\mathfrak{N}_{\alpha'}(y, \eta'; \rho) = \rho^{-\sigma|\alpha'|} \eta^{\alpha'} / \alpha'! + \sum_{\beta' < \alpha'} \rho^{-\sigma|\beta'|} b_{\alpha', \beta'}(y; \rho) \eta^{\beta'}, \\
&|b_{\alpha', \beta'(\tilde{\beta})}(y; \rho)| \le C_{\alpha', \beta', \tilde{\beta}} \rho^{-\sigma(|\alpha'| - |\beta'| - [(|\alpha'| - |\beta'|)/2])}, \\
&b_{\alpha', \beta'}(y; \rho) \equiv 0 \quad \text{if } |\alpha'| - |\beta'| = 1 \text{ and } \beta' < \alpha'
\end{aligned}$$

for $y \in V$, where $[a]$ denotes the largest integer $\le a$. Now we make an asymptotic change of variables, again:

$$z_1 = \rho^{\sigma_1}(y_1 - \hat{y}_1), \quad z'' = \rho^{\sigma/3}(y'' - \hat{y}''),$$

where $\sigma/2 < \sigma_1 < \sigma$. Put $y(z; \rho) = \hat{y} + (\rho^{-\sigma_1} z_1, \rho^{-\sigma/3} z'')$ and $\varphi(z; \rho) = \varphi(y(z; \rho))$. A simple calculation yields

$$\begin{aligned}
(3) \qquad &\Phi(\hat{y}, \rho^{-\sigma+\sigma_1} D_1, \rho^{-2\sigma/3} D'''; \hat{\gamma})(\exp[i\rho^\sigma \varphi(z; \rho)] v(z)) \\
&= \exp[i\rho^\sigma \varphi(z; \rho)] \sum_{|\alpha'| \ge p} \Phi^{(\alpha')}(\hat{y}, (\nabla_{y'} \varphi)(y(z; \rho); \hat{\gamma}) \\
&\quad \times \{\rho^{-\sigma|\alpha'| + \sigma_1 \alpha_1 + \sigma|\alpha'''|/3} D^{\alpha'} v(z) / \alpha'! \\
&\quad + \sum_{\beta' < \alpha'} \rho^{-\sigma|\beta'| + \sigma_1 \beta_1 + \sigma|\beta'''|/3} b_{\alpha', \beta'}(y(z; \rho); \rho) D^{\beta'} v(z)\} \\
&= \exp[i\rho^\sigma \varphi(z; \rho)] \rho^{-p(\sigma - \sigma_1)} \{\Phi^{(re_1')}(\hat{y}, \tau(\eta^{0'''}), \eta^{0'''}; \hat{\gamma}) D_1^p v(z) / p!
\end{aligned}$$

$$
\begin{aligned}
&+\rho^{-\delta_1}\sum_{|\alpha'|\le p_0} c_{\alpha'}(z;\rho)D^{\alpha'}v(z)\},\\
&|c_{\alpha'(\beta)}(z;\rho)|\le C_{\alpha',\beta}
\end{aligned}
$$

for $z\in\mathbb{R}^n$ with $y(z;\rho)\in V$, where $p_0=\deg_\eta\Phi(\hat{y},\eta';\hat{\gamma})$ and $\delta_1=\min\{\sigma/3,\sigma-\sigma_1,2\sigma_1-\sigma,\sigma_1-\sigma/3\}$. Indeed, we have

$$
\begin{aligned}
&p(\sigma-\sigma_1)-\sigma|\alpha'|+\sigma_1\beta_1+\sigma|\beta'''|/3+\sigma[(|\alpha'|-|\beta'|)/2]\\
&\le -(|\alpha'|-p)(\sigma-\sigma_1)-(|\alpha'|-|\beta'|)(\sigma_1-\sigma/2)-(\sigma_1-\sigma/3)|\beta'''|\\
&\le -(2\sigma_1-\sigma)\quad \text{if } |\alpha'|\ge p,\ \beta'<\alpha' \text{ and } |\beta'|\le|\alpha'|-2,\\
&p(\sigma-\sigma_1)-\sigma|\alpha'|+\sigma_1\alpha_1+\sigma|\alpha'''|/3\\
&=-(|\alpha'|-p)(\sigma-\sigma_1)-(\sigma_1-\sigma/3)|\alpha'''|.
\end{aligned}
$$

Put $E(z,\rho)=\exp[i\hat{\gamma}\rho^{1-\sigma/3}z_n]$ and

$$
\begin{aligned}
\widetilde{P}_\rho(z,\zeta)&=P(x^0+\rho^{-\delta q}y(z;\rho),\rho^{\delta q}(\rho^{\sigma_1}\zeta_1,\rho^{\sigma/3}\zeta''))\\
&\qquad(=P_\rho(y(z;\rho),\rho^{\sigma_1}\zeta_1,\rho^{\sigma/3}\zeta'')).
\end{aligned}
$$

Then we can write

$$
\widetilde{P}_\rho(z,\zeta)=\widetilde{Q}_N(z,\rho^{\sigma_1}\zeta_1,\rho^{\sigma/3}\zeta'';\rho)+\rho^{-N}\widetilde{R}_N(z,\rho^{\sigma_1}\zeta_1,\rho^{\sigma/3}\zeta'';\rho).
$$

Here $\widetilde{R}(z,\zeta;\rho)=\sum_{|\alpha|\le m}\widetilde{R}_{N,\alpha}(z;\rho)\zeta^\alpha$ and for any $W\Subset\mathbb{R}^n$ there are $C_{N,W,\beta}>0$ ($\beta\in(\mathbb{Z}_+)^n$) such that

$$
|\widetilde{R}_{N,\alpha(\beta)}(z;\rho)|\le C_{N,W,\beta}\quad \text{for } z\in W \text{ and } \rho\ge 1.
$$

Let W be an open neighborhood of 0 in $\mathbb{R}^n$, and choose $\rho(W)\ge 1$ so that $y(z;\rho)\in V$ for $z\in W$ and $\rho\ge\rho(W)$. Then we have

$$
\begin{aligned}
&\widetilde{P}_\rho(z,D)(E(z;\rho)\exp[i\rho^\sigma\varphi(z;\rho)]v(z))\\
&=E(z;\rho)\{\widetilde{Q}_N(z,\hat{\gamma}\rho e_n+(\rho^{\sigma_1}D_1,\rho^{\sigma/3}D'');\rho)\\
&\qquad+\rho^{-N}\widetilde{R}_N(z,\hat{\gamma}\rho e_n+(\rho^{\sigma_1}D_1,\rho^{\sigma/3}D'');\rho)\}(\exp[i\rho^\sigma\varphi(z;\rho)]v(z))
\end{aligned}
$$

for $z\in W$ and $\rho\ge\rho(W)$. By (1) we have

$$
\widetilde{Q}_N(z,\hat{\gamma}\rho e_n+\rho^\sigma\eta;\rho)=\hat{\gamma}^{m-r}\rho^{\mu_0}\{\Phi(\hat{y},\eta';\hat{\gamma})+\rho^{-\delta_0'}\tilde{r}_N(z,\eta;\rho)\},
$$

where $\delta_0'=\min\{\delta_0,\sigma/3\}$. Since $0<\sigma_1<\sigma$, this, together with (3), gives

$$
\widetilde{Q}_N(z,\hat{\gamma}\rho e_n+(\rho^{\sigma_1}D_1,\rho^{\sigma/3}D'');\rho)(\exp[i\rho^\sigma\varphi(z;\rho)]v(z))
$$

$$= \exp[i\rho^\sigma\varphi(z;\rho)]\hat{\gamma}^{m-r}\rho^{-p(\sigma-\sigma_1)+\mu_0}\{a_0D_1^pv(z)/p! \\ +\rho^{-\delta_1}\sum_{|\alpha'|\le p_0} c_{\alpha'}(z;\rho)D^{\alpha'}v(z)+\rho^{-\delta_0'+p(\sigma-\sigma_1)}\hat{r}(z,D;\rho)v(z)\}$$

for $z\in W$ and $\rho\ge\rho(W)$, where $a_0=\Phi^{(re_1')}(\hat{y},\tau(\eta^{0'''}),\eta^{0'''})$ $(\neq 0)$. Here $\hat{r}(z,\zeta;\rho)$ is a polynomial of ζ of degree m and has the same properties for $\rho\ge\rho(W)$ as $\widetilde{R}_N(z,\zeta;\rho)$. Now let us choose $N\in\mathbb{N}$ and $\sigma_1>0$ so that

$$N\ge p(\sigma-\sigma_1)-\mu_0+m+\delta_1,$$
$$\sigma_1=\max\{2\sigma/3,\sigma-\delta_0'/(p+1)\}.$$

From Lemma 2 it follows that there are $\ell\in\mathbb{Z}_+$ and $C>0$ such that

$$|u(0)|\le C\sup_{|\beta|\le\ell}\sup_{z_1\le 0}\rho^{(\delta q_1+\sigma_1)\beta_1+\delta\langle q'',\beta''\rangle+\sigma|\beta''|/3}|D^\beta\widetilde{P}_\rho(z,D)u(z)| \tag{4}$$

for $u\in C_0^\infty(W)$ and $\rho\ge\rho(W)$. Since $\delta_1\le\sigma-\sigma_1\le\delta_0'-p(\sigma-\sigma_1)$, we have

$$\widetilde{P}_\rho(z,D)(E(z;\rho)\exp[i\rho^\sigma\varphi(z;\rho)]v(z))=E(z;\rho)\exp[i\rho^\sigma\varphi(z;\rho)] \\ \times\hat{\gamma}^{m-r}\rho^{-p(\sigma-\sigma_1)+\mu_0}\{a_0D_1^pv(z)/p!-\rho^{-\delta_1}H(z,D;\rho)v(z)\}$$

for $z\in W$ and $\rho\ge\rho(W)$, where $H(z,\zeta;\rho)$ is a polynomial of ζ of degree m and has the same properties as $\hat{r}(z,\zeta;\rho)$. We define $\{u_j(z;\rho)\}_{j=0,1,2,\cdots}$ by

$$\begin{cases} u_0(z;\rho)=1, \\ D_1^pu_j(z;\rho)=a_0^{-1}H(z,D;\rho)u_{j-1}(z;\rho), \\ D_1^ku_j(0,z'';\rho)=0 \quad (0\le k\le p-1), \\ (j\ge 1), \end{cases}$$

for $z\in W$ and $\rho\ge\rho(W)$. Note that

$$|D^\beta u_j(z;\rho)|\le C_{j,\beta}\quad\text{for } z\in W,\ \rho\ge\rho(W)\text{ and }\beta\in(\mathbb{Z}_+)^n.$$

It is easy to see that

$$\varphi(z;\rho)=\rho^{-\sigma/3}z'''\cdot\eta^{0'''}+i\rho^{-2\sigma/3}|z''|^2+\rho^{-\sigma_1}\tau(\eta^{0'''})z_1+O(\rho^{-\sigma_1-\sigma/3}) \\ \text{as } \rho\to\infty,$$
$$\operatorname{Im}\ \rho^\sigma\varphi(z;\rho)\ge(\rho^{\sigma-\sigma_1}\operatorname{Im}\ \tau(\eta^{0'''})z_1+\rho^{\sigma/3}|z''|^2)/2$$

for $z\in W$ with $z_1\le 0$ and $\rho\ge\rho(W)$, modifying $\rho(W)$ if necessary. Let $\chi(z)$ be a function in $C_0^\infty(W)$ such that $\chi(z)=1$ near 0, and put

$$u_M(z;\rho)=\sum_{j=0}^{[M-1]}\rho^{-\delta_1 j}u_j(z;\rho)\chi(z)$$

for $\rho \geq \rho(W)$. Then, by standard arguments we have

$$\begin{aligned}&\sup_{z_1\leq 0,|\beta|\leq \ell} \rho^{(\delta q_1+\sigma_1)\beta_1+\delta\langle q'',\beta''\rangle+\sigma|\beta''|/3}\\ &\qquad\times |D^\beta\{\widetilde{P}_\rho(z,D)(E(z;\rho)\exp[i\rho^\sigma\varphi(z;\rho)]u_M(z;\rho))\}|\\ &\leq C_M\rho^{\nu(\ell)-M\delta_1}\leq C_M\rho^{-1}\quad \text{if } M\delta_1\geq 1+\nu(\ell) \text{ and } \rho\geq\rho(W),\end{aligned}$$

where $\nu(\ell) = -p(\sigma-\sigma_1)+\delta(1+q_1)(r+\ell)+\sigma r+\sigma_1\ell+(1+\delta q_n)(m-r)$. On the other hand, $u_M(0;\rho)=1$, which contradicts (4). This proves Theorem 1.

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