On the Cauchy Problem for Hyperbolic Operators with Double Characteristics whose Principal Parts Have Time Dependent Coefficients

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Abstract

In this paper we investigate the Cauchy problem for hyperbolic operators with double characteristics in the framework of the space of C^∞ functions. In the case where the coefficients of their principal parts depend only on the time variable and are real analytic, we give a sufficient condition for C^∞ well-posedness, which is also a necessary one when the space dimension is less than 3 or the coefficients of the principal parts are semi-algebraic functions (*e.g.*, polynomials) of the time variable.

1. Introduction

We say that a (partial differential) operators has time dependent coefficients if the coefficients depend only on the time variable. In [13] we studied the Cauchy problem for hyperbolic operators with double characeristics which have time dependent coefficients, and gave sufficient conditions for the Cauchy problem to be C^∞ well-posed, assuming that the coefficients of the principal parts are real analytic functions of the time variable.

In this paper we shall study the Cauchy problem for hyperbolic operators with double characteristics in the case where the principal parts have time dependent coefficients and the coefficients of the lower order terms depend on

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both the time variable and the space variables. And we shall give sufficient conditions for C^∞ well-posedness under the assumptions that the coefficients of the principal parts and the subprincipal symbols are real analytic in the time variable. These conditions are also necessary conditions if the space dimension is less than 3, or if the coefficients of the principal parts and the subprincipal symbols are semi-algebraic functions (*e.g.*, polynomials) with respect to the time variable. Our results are extensions of the results given in [12] to higher-order hyperbolic operators. For some examples and further literatures we refer to [12] and [13].

Let $m \in \mathbf{N}$ and $P(t,x,\tau,\xi) \equiv \tau^m + \sum_{j=1}^m \sum_{|\alpha|\le j} a_{j,\alpha}(t,x)\tau^{m-j}\xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1,\cdots,\xi_n)$ of degree m whose coefficients $a_{j,\alpha}(t,x)$ belong to $C^\infty([0,\infty)\times\mathbf{R}^n)$. Here $\alpha = (\alpha_1,\cdots,\alpha_n) \in (\mathbf{Z}_+)^n$ is a multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, where $\mathbf{Z}_+ = \mathbf{N}\cup\{0\}$ ($= \{0,1,2,3,\cdots\}$). We consider the Cauchy problem

$$\text{(CP)}\qquad \begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty)\times\mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (\ 0\le j\le m-1) \end{cases}$$

in the framework of the space of C^∞ functions, where $D_t = -i\partial/\partial t$ ($= -i\partial_t$), $D_x = (D_1,\cdots,D_n) = -i(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $f(t,x) \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j(x) \in C^\infty(\mathbf{R}^n)$ ($0\le j\le m-1$).

Definition 1.1. The Cauchy problem (CP) is said to be C^∞ well-posed if the following conditions (E) and (U) are satisfied:

(E) For any $f \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)$ ($0\le j\le m-1$) there is $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfying (CP).

(U) If $s > 0$, $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$, $D_t^j u(t,x)|_{t=0} = 0$ ($0\le j\le m-1$) and $P(t,x,D_t,D_x)u(t,x)$ vanishes for $t < s$, then $u(t,x)$ also vanishes for $t < s$.

We assume throughout the paper that

(A-1) $a_{j,\alpha}(t,x) \equiv a_{j,\alpha}(t)$ and $a_{j,\alpha}(t)$ is real analytic in $[0,\infty)$ if $1\le j\le m$ and $|\alpha| = j$.

From (A-1) there are a complex neighborhood Ω of $[0,\infty)$ (in $\mathbf{C}$) and $\delta_0 > 0$ such that $[-\delta_0,\infty)\subset\Omega$, $\Omega\cap\{\lambda\in\mathbf{C};\ \mathrm{Re}\,\lambda\le T\}$ is compact for any $T > 0$ and $a_{j,\alpha}(t)$ ($1\le j\le m$, $|\alpha| = j$) are regarded as analytic functions defined in Ω. Put

$$p(t,\tau,\xi) = \tau^m + \sum_{j=1}^m a_j^0(t,\xi)\tau^{m-j}\ \ (= P_m(t,\tau,\xi)),$$

$$a_j^0(t,\xi) = \sum_{|\alpha|=j} a_{j,\alpha}(t)\xi^\alpha,$$

$$P_k(t,x,\tau,\xi) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t,x)\tau^{m-j}\xi^\alpha \quad (\ 0 \le k \le m-1).$$

We also assume that the following conditions are satisfied:

(H) $p(t,\tau,\xi)$ is hyperbolic with respect to $\vartheta = (1,0,\cdots,0) \in \mathbf{R}^{n+1}$ for $t \in [-\delta_0,\infty)$, *i.e.*,

$$p(t,\tau - i,\xi) \neq 0 \quad \text{for any } (t,\tau,\xi) \in [-\delta_0,\infty) \times \mathbf{R} \times \mathbf{R}^n.$$

(A-2) $a_{j,\alpha}(t,x) \in C^\infty([-\delta_0,\infty) \times \mathbf{R}^n)$ ($1 \le j \le m$, $|\alpha| = j-1$), and for any $R > 0$ there are $C_R > 0$ and $A_R > 0$ such that

$$|\partial_t^k a_{j,\alpha}(t,x)| \le C_R A_R^k k!$$
$$\text{if } 1 \le j \le m,\ |\alpha| = j-1,\ k \in \mathbf{Z}_+,\ t \in [-\delta_0,R],\ x \in \mathbf{R}^n \text{ and } |x| \le R$$

(D) The characteristic roots are at most double, *i.e.*,

$$\partial_\tau^2 p(t,\tau,\xi) \neq 0$$
$$\text{if } (t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times S^{n-1} \text{ and } p(t,\tau,\xi) = \partial_\tau p(t,\tau,\xi) = 0,$$

where $S^{n-1} = \{\xi \in \mathbf{R}^n;\ |\xi| = 1\}$.

Note that the assumption (A-2) is satisfied if $a_{j,\alpha}(t,x)$ ($1 \le j \le m$, $|\alpha| = j-1$) are real analytic in $[-\delta_0,\infty) \times \mathbf{R}^n$. Let $\Gamma(p(t,\cdot,\cdot),\vartheta)$ be the connected component of the set $\{(\tau,\xi) \in \mathbf{R}^{n+1} \setminus \{0\};\ p(t,\tau,\xi) \neq 0\}$ which contains ϑ, and define the genralized flows $K^\pm_{(t_0,x^0)}$ for $p(t,\tau,\xi)$ by

$$\begin{aligned} K^\pm_{(t_0,x^0)} = \{(t(s),x(s)) \in [0,\infty) \times \mathbf{R}^n;\ &\pm s \ge 0 \text{ and } \{(t(s),x(s))\} \text{ is} \\ &\text{a Lipschitz continuous curve in } [0,\infty) \times \mathbf{R}^n \text{ satisfying} \\ &(d/ds)(t(s),x(s)) \in \Gamma(p(t,\cdot,\cdot),\vartheta)^* \ (\ a.e.\ s) \text{ and} \\ &(t(0),x(0)) = (t_0,x^0)\}, \end{aligned}$$

where $(t_0,x^0) \in [0,\infty) \times \mathbf{R}^n$ and $\Gamma^* = \{(t,x) \in \mathbf{R}^{n+1};\ t\tau + x \cdot \xi \ge 0$ for any $(\tau,\xi) \in \Gamma\}$. $K^+_{(t_0,x^0)}$ (resp. $K^-_{(t_0,x^0)}$) gives an estimate of the influence domain (resp. the dependence domain) of (t_0,x^0) (see Theorem 1.2 below). To describe conditions on the lower order terms we define the polynomials $h_j(t,\tau,\xi)$ of (τ,ξ) by

$$|p(t,\tau - i\gamma,\xi)|^2 = \sum_{j=0}^{m} \gamma^{2j} h_{m-j}(t,\tau,\xi)$$

for $(t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times \mathbf{R}^n$ and $\gamma \in \mathbf{R}$.

Since $|p(t,\tau - i\gamma,\xi)|^2 = \prod_{j=1}^m ((\tau - \lambda_j(t,\xi))^2 + \gamma^2)$, we have

$$(1.1) \qquad h_k(t,\tau,\xi) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} \prod_{l=1}^k (\tau - \lambda_{j_l}(t,\xi))^2 \quad (\, 1 \le k \le m),$$

where $p(t,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,\xi))$. Let $\mathcal{R}(\xi)$ be a set-valued function, whose values are discrete subsets of $\mathbf{C}$, defined for $\xi \in S^{n-1}$ satisfying the following:

For any $T > 0$ there is $N_T \in \mathbf{Z}_+$ such that

$$(1.2) \qquad \#\{\lambda \in \mathcal{R}(\xi);\ \mathrm{Re}\,\lambda \in [0,T]\} \le N_T \quad \text{for } \xi \in S^{n-1}.$$

Here $\#A$ denotes the number of the elements of a set A. The following condition is corresponding to a so-called Levi condition:

(L) For any $T > 0$ and $x \in \mathbf{R}^n$ there is $C > 0$ satisfying

$$\min\Big\{ \min_{s \in \mathcal{R}(\xi)} |t - s|, 1 \Big\} |sub\ \sigma(P)(t,x,\tau,\xi)| \le C h_{m-1}(t,\tau,\xi)^{1/2}$$
$$\text{for } (t,\tau,\xi) \in [0,T] \times \mathbf{R} \times S^{n-1},$$

where $\min_{s \in \mathcal{R}(\xi)} |t - s| = 1$ if $\mathcal{R}(\xi) = \emptyset$.

Here $sub\ \sigma(P)(t,x,\tau,\xi)$ denotes the subprincipal symbol of $P(t,x,D_t,D_x)$, *i.e.*,

$$sub\ \sigma(P)(t,x,\tau,\xi) = P_{m-1}(t,x,\tau,\xi) + \frac{i}{2}\partial_t \partial_\tau p(t,\tau,\xi).$$

Concerning sufficiency of C^∞ well-posedness, we have the following

Theorem 1.2. *Assume that the conditions* (A-1), (A-2), (H), (D) *and* (L) *are satisfied. Then the Cauchy problem* (CP) *is* C^∞ *well-posed. Moreover, if* $(t_0, x^0) \in (0,\infty) \times \mathbf{R}^n$ *and* $u \in C^\infty([0,\infty) \times \mathbf{R}^n)$ *satisfies* (CP), $u_j(x) = 0$ *near* $\{x \in \mathbf{R}^n; (0,x) \in K^-_{(t_0,x^0)}\}$ $(\, 0 \le j \le m-1)$ *and* $f = 0$ *near* $K^-_{(t_0,x^0)}$ *(in* $[0,\infty) \times \mathbf{R}^n$*), then* $(t_0,x^0) \notin \mathrm{supp}\, u$.

Remark. (i) The condition (L) depends on the choice of a set-valued function $\mathcal{R}(\xi)$. However, if (L) is satisfied for a set-valued function $\mathcal{R}(\xi)$, then $(\mathrm{L})_0$, which is defined in Theorem 1.3 below, is also satisfied (see the remark of Theorem 1.3). (ii) If $m \le 2$, then the theorem is valid under the assumptions (A-1), (H)$'$, (D) and (L), where the condition (H)$'$ is defined below (see [12]).

Next we shall give results on necessity for C^∞ well-posedness. Instead of the condition (H) we assume that

(H)′ $p(t, \tau, \xi)$ is hyperbolic with respect to ϑ for $t \in [0, \infty)$.

Write

$$
\begin{aligned}
p(t,\tau,\xi) &= \prod_{j=1}^{m} (\tau - \lambda_j(t,\xi)), \\
\mu_{j,k}(t,\xi) &= (\lambda_j(t,\xi) - \lambda_k(t,\xi))^2,
\end{aligned}
$$

where $\lambda_1(t,\xi) \le \lambda_2(t,\xi) \le \cdots \le \lambda_m(t,\xi)$. Define $\{D_\mu(t,\xi)\}_{1 \le \mu \le M}$ by

$$
\prod_{1 \le j < k \le m} (\tau - \mu_{j,k}(t,\xi)) = \tau^M + \sum_{l=1}^{M} (-1)^l D_l(t,\xi) \tau^{M-l},
$$

where $M = \binom{m}{2}$. Note that $D_M(t,\xi)$ $(\equiv D(t,\xi))$ is the discriminant of $p(t,\tau,\xi) = 0$ in τ. Putting $D_0(t,\xi) \equiv 1$, for each $\xi \in S^{n-1}$ there is $r(\xi) \in \mathbf{Z}_+$ such that $0 \le r(\xi) \le M$ and

$$
\begin{aligned}
&D_M(t,\xi) \equiv \cdots \equiv D_{M-r(\xi)+1}(t,\xi) \equiv 0 \text{ in } t, \\
&D_{M-r(\xi)}(t,\xi) \not\equiv 0 \text{ in } t.
\end{aligned}
$$

We define

$$
\mathcal{R}_0(\xi) = \{(\operatorname{Re}\lambda)_+;\ \lambda \in \Omega,\ D_{M-r(\xi)}(\lambda,\xi) = 0\} \quad \text{for } \xi \in S^{n-1},
$$

where $a_+ = \max\{0, a\}$ for $a \in \mathbf{R}$. By Lemma 2.2 below we may assume that for any $T > 0$ there is $N_T \in \mathbf{Z}_+$ satisfying

$$
\#(\mathcal{R}_0(\xi) \cap [0,T]) \le N_T \quad \text{for } \xi \in S^{n-1},
$$

modifying Ω if necessary. Let U be a semi-algebraic set in $\mathbf{R}$, and let $h(t)$ be a function defined in U. For the definition of semi-algebraic sets we refer to [14], for example. We say that $h(t)$ is semi-algebraic in U if the graph $\{(t, h(t)) \in \mathbf{R}^2;\ t \in U\}$ is a semi-algebraic set. For basic properties of semi-algebraic functions we refer to [14] and [15].

Theorem 1.3. *Assume that the condition* (A-1), (A-2), (H)′ *and* (D) *are satisfied. Moreover, we assume that the* $a_{j,\alpha}(t,x)$ ($1 \le j \le m$, $|\alpha| = j, j-1$) *are semi-algebraic in* $[0,\infty)$ *for each* $x \in \mathbf{R}^n$ *when* $n \ge 3$. *Then the condition*

(L)$_0$ *for any* $T>0$ *and* $x \in \mathbf{R}^n$ *there is* $C>0$ *such that*

$$\min\Big\{\min_{s\in\mathcal{R}_0(\xi)}|t-s|,1\Big\}|sub\ \sigma(P)(t,x,\tau,\xi)| \le Ch_{m-1}(t,\tau,\xi)^{1/2}$$
$$\text{for } (t,\tau,\xi)\in[0,T]\times\mathbf{R}\times S^{n-1}$$

is satisfied if the Cauchy problem (CP) *is* C^∞ *well-posed.*

Remark. (i) We directly prove that the condition (L)$_0$ is satisfied if the condition (L) is satisfied (see Lemma 4.1). (ii) If $m \le 2$ and $n \le 2$, then the theorem is valid under the assumptions (A-1), (H)$'$ and (D) (see [12]). Moreover, if $m \le 2$ and $n \ge 3$, the theorem is valid under the assumptions (A-1), (H)$'$, and (D) and the assumption that $a_{j,\alpha}(t,x)$ ($0 \le j \le m-1$, $|\alpha| = j$) are semi-algebraic in $[0,\infty)$ for each $x \in \mathbf{R}^n$.

The remainder of this paper is organized as follows. §2 and §3 will be divided into subsections. In §2 we shall prove Theorem 1.2. Theorem 1.3 will be proved in §3. In §4 we shall give some remarks.

2. Proof of Theorem 1.2

In this section we shall give the proof of Theorem 1.2, deriving microlocal energy estimates. To obtain local energy estimates from microlocal ones we shall adopt ideas used in [7], although we can not directly use the results in [7]. Assume that the conditions (A-1), (A-2), (H), (D) and (L) are satisfied.

2.1. Preliminaries

Let U be an open set in $\mathbf{R}^n$, and let $a(t,\xi)$ be a real analytic function defined in $[-\delta,\delta]\times\overline{U}$, where $\delta>0$. Lemma 2.2 below is a key lemma. To prove Lemma 2.2 we need the following

Lemma 2.1. *Let* S *be a subset of* $(\mathbf{Z}_+)^n$, *and let* $\beta^0 \in S$. *Assume that there is* $\beta^1 \in S$ *satisfying* $\beta^0 \not\le \beta^1$, *i.e., there is* $k \in \mathbf{N}$ *with* $k \le n$ *such that* $\beta^0_k > \beta^1_k$ *Then there are* $\nu^0 \in (\mathbf{Z}_+)^n$ *and* $\alpha^0 \in S$ *such that* $\alpha^0 \ne \beta^0$ *and*

$$\alpha^0\cdot\nu^0 < \alpha\cdot\nu^0 \quad \text{for } \alpha \in S\setminus\{\alpha^0\}. \tag{2.1}$$

Proof. Let us prove the lemma by induction. If $n=1$, then, choosing $\alpha^0(=\alpha^0_1) = \min_{\alpha\in S}\alpha$ ($\in \mathbf{Z}_+$) and $\nu^0 = 1$ ($\in \mathbf{Z}_+$), we can show that the lemma is valid. Let $l \in \mathbf{N}$, and suppose that the lemma is valid when $n=l$.

Let $n = l + 1$. By assumption on S there are $\beta^1 \in S$ and $k \in \mathbf{N}$ with $k \le n$ such that $\beta^0_k > \beta^1_k$. We may assume that $k = n$, *i.e.*, $\beta^0_n > \beta^1_n$. Put

$$S_1 = \{\alpha \in S;\ \alpha_n = \min_{\beta \in S} \beta_n\ (< \beta^0_n)\}.$$

Note that $\beta^0 \notin S_1$. Let us first consider the case where there is $\alpha^0 \in S_1$ such that $\alpha^0 \le \alpha$ for any $\alpha \in S_1$. If we chose $\nu^0 = (1, \cdots, 1, l) \in (\mathbf{Z}_+)^n$, with $l = \sum_{k=1}^{n-1} \alpha^0_k + 1$, then $\alpha^0 \ne \beta^0$ and (2.1) is satisfied. Indeed, it is obvious that $\alpha^0 \cdot \nu^0 < \alpha \cdot \nu^0$ for $\alpha \in S_1 \setminus \{\alpha^0\}$. For $\alpha \in S \setminus S_1$ we have

$$\alpha \cdot \nu^0 \ge l\alpha^0_n + l > l\alpha^0_n + \sum_{k=1}^{n-1} \alpha^0_k = \alpha^0 \cdot \nu^0.$$

Next consider the case where for any $\beta \in S_1$ there is $\alpha \in S_1$ satisfying $\beta \not\le \alpha$. Fix $\tilde\beta^0 \in S_1$. Then there is $\tilde\beta^1 \in S_1$ such $\tilde\beta^0 \not\le \tilde\beta^1$. We write $\alpha' = (\alpha_1, \cdots, \alpha_{n-1})$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$, and put $S'_1 = \{\alpha' \in (\mathbf{Z}_+)^{n-1};\ \alpha \in S_1\}$. Then we have $S'_1 \subset (\mathbf{Z}_+)^{n-1}$, $\tilde\beta^{0\prime}, \tilde\beta^{1\prime} \in S'_1$ and $\tilde\beta^{0\prime} \not\le \tilde\beta^{1\prime}$. So, by induction assumption there are $\tilde\nu^{0\prime} \in (\mathbf{Z}_+)^{n-1}$ and $\tilde\alpha^{0\prime} \in S'_1$ such that $\tilde\alpha^{0\prime} \ne \tilde\beta^{0\prime}$ and

$$\tilde\alpha^{0\prime} \cdot \tilde\nu^{0\prime} < \alpha' \cdot \tilde\nu^{0\prime} \quad \text{for } \alpha' \in S'_1 \setminus \{\tilde\alpha^{0\prime}\}.$$

Taking $\nu^0 = (\tilde\nu^{0\prime}, l)$, $l = \tilde\alpha^{0\prime} \cdot \tilde\nu^{0\prime} + 1$ and $\alpha^0 = (\tilde\alpha^{0\prime}, \tilde\beta^0_n)$, we have

$$\begin{aligned} &\alpha^0 \cdot \nu^0 < \alpha \cdot \nu^0 \quad \text{for } \alpha \in S_1 \setminus \{\alpha^0\}, \\ &\alpha \cdot \nu^0 \ge l\tilde\beta^0_n + l > \alpha^0 \cdot \nu^0 \quad \text{for } \alpha \in S \setminus S_1. \end{aligned}$$

This proves the lemma. □

Put

$$\kappa(\xi) = \int_0^\delta |a(t,\xi)|^2\, dt\ (\ge 0).$$

Lemma 2.2. *There are $m_0 \in \mathbf{N}$ and $C_k > 0$ ($k \in \mathbf{Z}_+$) such that for any $\xi \in \overline{U}$ there are $m(\xi) \in \mathbf{Z}_+$ and $a_k(\xi) \in \mathbf{R}$ ($1 \le k \le m(\xi)$) satisfying $m(\xi) \le m_0$ and*

$$\begin{aligned} &C_0^{-1}\sqrt{\kappa(\xi)}|t^{m(\xi)} + a_1(\xi)t^{m(\xi)-1} + \cdots + a_{m(\xi)}(\xi)| \le |a(t,\xi)| \le C_0\sqrt{\kappa(\xi)}, \\ &|\partial_t^k a(t,\xi)| \le C_k\sqrt{\kappa(\xi)} \quad (\ k \in \mathbf{Z}_+) \end{aligned}$$

for $t \in [-\delta, \delta]$, with a modification of δ if necessary.

Remark. (i) Let $\xi^0 \in \overline{U}$. It is obvious that $a(t,\xi^0) \not\equiv 0$ in t if and only if $\kappa(\xi^0) \ne 0$. So, if $\kappa(\xi^0) \ne 0$, then one can apply the Weierstrass preparation theorem to $a(t,\xi)$ at $(t,\xi)=(0,\xi^0)$ to prove the lemma with U replaced by a neighborhood of ξ^0. (ii) $a(t,\xi)$ is regarded as an analytic function defined in a complex neighborhood of $[-\delta,\delta]\times\overline{U}$. Then from Lemma 2.2 and its proof there is $\delta'>0$ satisfying

$$\#\{\lambda\in\mathbf{C};\ \mathrm{Re}\,\lambda\in[-\delta-\delta',\delta+\delta'],\ |\,\mathrm{Im}\,\lambda|\le\delta' \text{ and } a(\lambda,\xi)=0\}\le m_0$$
$$\text{if } \xi\in\overline{U} \text{ and } a(t,\xi)\not\equiv 0 \text{ in } t.$$

(iii) Assume that $a(t,\xi)\ge 0$ for $(t,\xi)\in[-\delta,\delta]\times\overline{U}$. Then we can prove the lemma with $\sqrt{\kappa(\xi)}$ replaced by

$$\tilde\kappa(\xi)=\int_0^\delta a(t,\xi)\,dt,$$

using $\tilde\kappa(\xi)$ instead of $\kappa(\xi)$ in the proof below.

Proof. If $\kappa(\xi)\equiv 0$, then the lemma becomes trivial. So we may assume that $\kappa(\xi)\not\equiv 0$. Let $\xi^0\in\overline{U}$. We apply Hironaka's resolution theorem to $\kappa(\xi)$ (see [1]). Then there are an open neighborhood $U(\xi^0)$ of ξ^0, a real analytic manifold $\widetilde{U}(\xi^0)$, a proper analytic mapping $\varphi\equiv\varphi(\xi^0)$: $\widetilde{U}(\xi^0)\ni\tilde u\mapsto\varphi(\tilde u)(\equiv\varphi(\tilde u;\xi^0))\in U(\xi^0)$ satisfying the following:

(i) φ: $\widetilde{U}(\xi^0)\setminus\widetilde{A}\to U(\xi^0)\setminus A$ is an isomorphism, where $A=\{\xi\in\overline{U};\ \kappa(\xi)=0\}$ and $\widetilde{A}=\varphi^{-1}(A)$.

(ii) For each $p\in\widetilde{U}(\xi^0)$ there are local analytic coordinates $X\,(\equiv X^p)=(X_1,\cdots,X_n)\,(=(X_1^p,\cdots,X_n^p))$ centered at p, $r(p)\in\mathbf{Z}_+$ with $r(p)\le n$, $s_k(p)\in\mathbf{N}$ ($1\le k\le r(p)$), a neighborhood $\widetilde{U}(\xi^0;p)$ of p and a real analytic function $e(X)$ in $\widetilde{V}(\xi^0;p)$ such that $e(X)>0$ for $X\in\widetilde{V}(\xi^0;p)$ and

$$\kappa(\varphi(\tilde u))=e(X(\tilde u))\prod_{k=1}^{r(p)}X_k(\tilde u)^{2s_k(p)}\quad(\ \tilde u\in\widetilde{U}(\xi^0;p)),\tag{2.2}$$

where $\widetilde{V}(\xi^0;p)=\{X(\tilde u);\ \tilde u\in\widetilde{U}(\xi^0;p)\}$ and $\prod_{k=1}^{r(p)}\cdots=1$ if $r(p)=0$.

Here $\widetilde{V}(\xi^0;p)$ is a neighborhood of 0 in $\mathbf{R}^n$ and we have used the fact that $\kappa(\xi)\ge 0$. Define $\tilde\varphi\,(\equiv\tilde\varphi(\xi^0,p)):\widetilde{V}(\xi^0;p)\to U(\xi^0)$ by $\tilde\varphi(X(\tilde u))\,(\equiv\tilde\varphi(X^p(\tilde u);\xi^0,p))=\varphi(\tilde u)\,(\equiv\varphi(\tilde u;\xi^0))$ for $\tilde u\in\widetilde{U}(\xi^0;p)$. Let $U_0(\xi^0)$ be a compact neighborhood of ξ^0 in $U(\xi^0)$, and put $\widetilde{U}_0(\xi^0)=\varphi^{-1}(U_0(\xi^0))$. Fix $p\in\widetilde{U}_0(\xi^0)$, and put

$$\alpha(p)=(s_1(p),\cdots,s_{r(p)}(p),0,\cdots,0)\in(\mathbf{Z}_+)^n.$$

We write $a(t, \tilde{\varphi}(X))$ as

$$(2.3) \qquad a(t, \tilde{\varphi}(X)) = \sum_{\alpha} c_\alpha(t;p) X^\alpha, \quad c_\alpha(t;p) = \frac{1}{\alpha!} \partial_X^\alpha a(t, \tilde{\varphi}(X))|_{X=0}$$

Put

$$S_p = \{\alpha \in (\mathbf{Z}_+)^n;\ c_\alpha(t;p) \not\equiv 0 \text{ in } t\}.$$

It follows from (2.2) that for $\nu = (\nu_1, \cdots, \nu_n) \in (\mathbf{Z}_+)^n$

$$(2.4) \qquad \int_0^\delta |a(t, \tilde{\varphi}(X))|^2 \, dt|_{X_k = s^{\nu_k}\ (1 \le k \le n)} = O(s^{2\alpha(p)\cdot\nu}) \quad \text{as } s \downarrow 0.$$

Suppose that there is $\beta^1 \in S_p$ satisfying $\alpha(p) \not\leq \beta^1$. Then Lemma 2.1 with $S = S_p \cup \{\alpha(p)\}$ and $\beta^0 = \alpha(p)$ implies that there are $\nu^0 \in (\mathbf{Z}_+)^n$ and $\alpha^0 \in S_p$ such that $\alpha^0 \neq \alpha(p)$ and

$$(2.5) \qquad \alpha^0 \cdot \nu^0 < \alpha \cdot \nu^0 \quad \text{for } \alpha \in S_p \cup \{\alpha(p)\} \setminus \{\alpha^0\}.$$

(2.4) and (2.5) with $\alpha \in S_p$ yield $\alpha^0 \cdot \nu^0 \geq \alpha(p) \cdot \nu^0$, which contradicts (2.5) with $\alpha = \alpha(p)$. Therefore, for $\alpha \in S_p$ we have $\alpha \geq \alpha(p)$. This, together with (2.2) and (2.3), gives $\alpha(p) \in S_p$. So we can write

$$(2.6) \qquad a(t, \tilde{\varphi}(X)) = X^{\alpha(p)}(c_{\alpha(p)}(t;p) + b(t, X;p)),$$

where $b(t, X; p)$ is real analytic in (t, X) and satisfies $b(t, 0; p) = 0$. Putting

$$a(t, X; p) = c_{\alpha(p)}(t;p) + b(t, X; p),$$

we can apply the Weierstrass preparation theorem to $a(t, X; p)$ at $(t, X) = (0, 0)$. Therefore, there are $\delta(p) > 0$, a neighborhood $\widetilde{V}(p)$ of 0 in $\widetilde{V}(\xi^0; p)$, $m(p) \in \mathbf{Z}_+$, a real analytic function $c(t, X; p)$ in $[-\delta(p), \delta(p)] \times \widetilde{V}(p)$ and real analytic functions $a_k(X; p)$ in $\widetilde{V}(p)$ ($1 \le k \le m(p)$) such that $c(t, X; p) \neq 0$ and

$$(2.7) \qquad a(t, X; p) = c(t, X; p)(t^{m(p)} + a_1(X; p)t^{m(p)-1} + \cdots + a_{m(p)}(X; p))$$

in $[-\delta(p), \delta(p)] \times \widetilde{V}(p)$. Note that $\delta(p)$, $\widetilde{V}(p)$, $m(p)$, $c(t, X; p)$ and the $a_k(X; p)$ also depend on ξ^0. Put $\widetilde{U}(p) = (X^p)^{-1}(\widetilde{V}(p))$ ($\subset \widetilde{U}(\xi^0; p)$). Since $\overline{U}$ is compact, there are $N \in \mathbf{N}$ and $\xi^j \in \overline{U}$ ($1 \le j \le N$), such that $\overline{U} \subset \bigcup_{j=1}^N \mathring{U}_0(\xi^j)$. Here $\mathring{A}$ denotes the interior of A ($\subset \mathbf{R}^n$). Since $\widetilde{U}_0(\xi^j)$ is compact, there are $P_j \in \mathbf{N}$ and $p^{j,k} \in \widetilde{U}_0(\xi^j)$ ($1 \le k \le P_j$) such that

$\widetilde{U}_0(\xi^j) \subset \bigcup_{k=1}^{P_j} \widetilde{U}(p^{j,k})$. Let $1 \le j \le N$ and $1 \le k \le P_j$. (2.2), (2.6) and (2.7) for $p = p^{j,k}$ give, with $C_0 > 0$,

$$\begin{aligned}&C_0^{-1}\sqrt{\kappa(\tilde{\varphi}(X;\xi^j,p^{j,k}))}|t^{m(p^{j,k})} + a_1(X;p^{j,k})t^{m(p^{j,k})-1} + \cdots + a_{m(p^{j,k})}(X;p^{j,k})| \\ &\quad \le |a(t,\tilde{\varphi}(X;\xi^j,p^{j,k}))| \le C_0\sqrt{\kappa(\tilde{\varphi}(X;\xi^j,p^{j,k}))}, \\ &|\partial_t^k a(t,\tilde{\varphi}(X;\xi^j,p^{j,k}))| \le C_k\sqrt{\kappa(\tilde{\varphi}(X;\xi^j,p^{j,k}))} \quad (k \in \mathbf{Z}_+)\end{aligned}$$

for $(t,X) \in [-\delta(p^{j,k}),\delta(p^{j,k})] \times \widetilde{V}(p^{j,k})$, which proves the lemma. □

Let $\kappa, \kappa' \in \mathbf{R}$, and let I be an interval of $\mathbf{R}$. We say that $a(t,x,\xi) \in S^\kappa_{\rho,\delta}(I \times T^*\mathbf{R}^n)$ if $a(t,x,\xi) \in C^\infty(I \times T^*\mathbf{R}^n)$ and

$$|D_t^j D_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \le C_{j,\alpha,\beta}\langle\xi\rangle^{\kappa-\rho|\alpha|+\delta|\beta|} \tag{2.8}$$

for $(t,x,\xi) \in I \times T^*\mathbf{R}^n$ and any $j \in \mathbf{Z}_+$ and $\alpha,\beta \in (\mathbf{Z}_+)^n$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $0 \le \rho,\delta \le 1$. When $a(t,x,\xi;\varepsilon)$ also depends on a parameter ε, we say that $a(t,x,\xi;\varepsilon) \in S^\kappa_{\rho,\delta}(I \times T^*\mathbf{R}^n)$ uniformly in ε if the $C_{j,\alpha,\beta}$ in (2.8) with $a(t,x,\xi)$ replaced by $a(t,x,\xi;\varepsilon)$ can be chosen so that they do not depend on ε. Moreover, we say that a symbol $a(t,x,\tau,\xi) \in \mathcal{S}^{\kappa,\kappa'}_{1,0}$ if $a(t,x,\tau,\xi) = \sum_{j=0}^{[\kappa]} a_j(t,x,\xi)\tau^j$ and the $a_j(t,x,\xi)$ are classical symbols and $a_j(t,x,\xi) \in S^{\kappa+\kappa'-j}_{1,0}(\mathbf{R} \times T^*\mathbf{R}^n)$, where $[\kappa]$ denotes the largest integer $\le \kappa$ and $\mathcal{S}^{\kappa,\kappa'}_{1,0} = \{0\}$ if $\kappa < 0$. We also write $\mathcal{S}^\kappa_{1,0} = \mathcal{S}^{\kappa,0}_{1,0}$ and $\mathcal{S}^{\kappa,-\infty}_{1,0} = \bigcap_{\kappa' \in \mathbf{R}} \mathcal{S}^{\kappa,\kappa'}_{1,0}$. When $a(t,x,\tau,\xi;\varepsilon) = \sum_{j=0}^{[\kappa]} a_j(t,x,\xi;\varepsilon)\tau^j$ depends on a parameter ε, we say that $a(t,x,\tau,\xi;\varepsilon) \in \mathcal{S}^{\kappa,\kappa'}_{1,0}$ uniformly on ε if the $a_j(t,x,\xi;\varepsilon)$ are classical symbols and $a_j(t,x,\xi;\varepsilon) \in S^{\kappa+\kappa'-j}_{1,0}(\mathbf{R} \times T^*\mathbf{R}^n)$ uniformly in ε. From the assumption (D) there are $\delta_1 > 0$, $N_0 \in \mathbf{N}$, open conic sets $\mathcal{C}_j$ and $\mathcal{C}_{j,0}$ in $\mathbf{R}^n \setminus \{0\}$, $r_j \in \mathbf{Z}_+$ ($1 \le j \le N_0$), $\tilde{p}_{j,k}(t,\tau,\xi) \in \mathcal{S}^2_{1,0}$ ($1 \le j \le N_0$, $1 \le k \le r_j$) and $\tilde{p}_{j,r_j+1}(t,\tau,\xi) \in \mathcal{S}^{m-2r_j}_{1,0}$ ($1 \le j \le N_0$) such that $2r_j \le m$, the $\tilde{p}_{j,k}(t,\tau,\xi)$ are positively homogeneous in (τ,ξ) for $|\xi| \ge 1/4$, $3\delta_1 \le \delta_0$, $\bigcup_{l=0}^{N_0} \mathcal{C}_{l,0} \supset S^{n-1}$, $\mathcal{C}_{j,0} \Subset \mathcal{C}_j$, and

$$p(t,\tau,\xi) = \prod_{k=1}^{r_j+1} \tilde{p}_{j,k}(t,\tau,\xi) \tag{2.9}$$

$$\text{for } (t,\tau,\xi) \in [-2\delta_1,4\delta_1] \times \mathbf{R} \times \overline{\mathcal{C}}_j \text{ with } |\xi| \ge 1/4,$$

$$\{\tau \in \mathbf{C};\ \tilde{p}_{j,k}(t,\tau,\xi) = 0\} \cap \{\tau \in \mathbf{C};\ \tilde{p}_{j,l}(t,\tau,\xi) = 0\} = \emptyset \tag{2.10}$$

$$\text{if } k \ne l,\ (t,\xi) \in [-2\delta_1,4\delta_1] \times \overline{\mathcal{C}}_j \text{ and } |\xi| \ge 1/4,$$

$$\partial_\tau \tilde{p}_{j,r_j+1}(t,\tau,\xi) \ne 0$$

$$\text{if } (t,\tau,\xi) \in [-2\delta_1,4\delta_1] \times \mathbf{R} \times \overline{\mathcal{C}}_j,\ |\xi| \ge 1/4 \text{ and } \tilde{p}_{j,r_j+1}(t,\tau,\xi) = 0$$

($1 \le j \le N_0$), where $\tilde{p}_{j,r_j+1}(t,\tau,\xi) = 1$ if $m - 2r_j = 0$ and $p(t,\tau,\xi) = \tilde{p}_{j,r_j+1}(t,\tau,\xi)$ if $r_j = 0$. Here $A \Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\overset{\circ}{B}$ of B for a bounded subset A and a subset B of $\mathbf{R}^n$. For conic sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbf{R}^n$ $\mathcal{C}_1 \Subset \mathcal{C}_2$ means that $\mathcal{C}_1 \cap S^{n-1} \Subset \mathcal{C}_2$. Denote by $p_{j,k}(t,\tau,\xi)$ the pricipal symbol of $\tilde{p}_{j,k}(t,\tau,\xi)$ ($1 \le j \le N_0$, $1 \le k \le r_j + 1$). So we have

$$\tilde{p}_{j,k}(t,\tau,\xi) = p_{j,k}(t,\tau,\xi) \quad \text{for } |\xi| \ge 1/4.$$

We assume that $p_{j,k}(t,\tau,\xi)$ ($1 \le k \le r_j$) are not strictly hyperbolic in τ for some $(t,\xi) \in [-2\delta_1, 4\delta_1] \times (\overline{\mathcal{C}}_j \cap S^{n-1})$, and that

$$\begin{aligned} &\{\tau \in \mathbf{C};\ (t,\xi) \in [-2\delta_1, 4\delta_1] \times (\overline{\mathcal{C}}_j \cap S^{n-1}),\ p_{j,k}(t,\tau,\xi) = 0\} \\ &\cap \{\tau \in \mathbf{C};\ (t,\xi) \in [-2\delta_1, 4\delta_1] \times (\overline{\mathcal{C}}_j \cap S^{n-1}),\ p_{j,l}(t,\tau,\xi) = 0\} = \emptyset \end{aligned}$$

for $1 \le k < l \le r_j + 1$, modifying δ_1 and $\mathcal{C}_j$ if necessary. Moreover, we can write

$$p_{j,k}(t,\tau,\xi) = (\tau - b_{j,k}(t,\xi))^2 - a_{j,k}(t,\xi) \quad (1 \le k \le r_j),$$

where $a_{j,k}(t,\xi) \ge 0$ and the $b_{j,k}(t,\xi)$ are real-valued. Choose $\Theta(t) \in \mathcal{E}^{\{3/2\}}(\mathbf{R})$ so that

$$\Theta(t) = \begin{cases} 1 & (t \le 3/2), \\ 0 & (t \ge 2). \end{cases}$$

Here $f(x) \in \mathcal{E}^{\{s\}}(D)$ ($\subset C^\infty(D)$) means that for any compact subset K of D there are positive constants C and A satisfying

$$|\partial^\alpha f(x)| \le C A^{|\alpha|} (\alpha!)^s \quad \text{for } \alpha \in (\mathbf{Z}_+)^n \text{ and } x \in K,$$

where D is an open subset of $\mathbf{R}^n$ and $s \ge 1$. For $h > 0$ we define $\Theta_h(t) = \Theta(t/h)$ and $\Theta_h(\xi) = \Theta_h(|\xi|)$. Choose $\rho(x) \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^n)$ so that $\operatorname{supp} \rho \subset \{x \in \mathbf{R}^n;\ |x| \le 1\}$, $\rho(x) \ge 0$ and $\int_{\mathbf{R}^n} \rho(x)\,dx = 1$. We put $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(\varepsilon^{-1} x)$, and define

$$a_{j,\alpha}(t,x;R,\varepsilon) = \Theta_{\delta_1}(-t) \int_{\mathbf{R}^n} \rho_\varepsilon(x-y) \Theta(|y| - R) a_{j,\alpha}(t,y)\,dy$$

for $(t,x) \in \mathbf{R}^{n+1}$, $R \ge 1$ and $\varepsilon \in (0,1]$ when $1 \le j \le m$ and $|\alpha| = j - 1$. Fix $R \ge 1$. It is easy to see that $a_{j,\alpha}(t,x;R,\varepsilon) \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$, $\operatorname{supp} a_{j,\alpha}(t,\cdot;R,\varepsilon) \subset \{x \in \mathbf{R}^n;\ |x| \le R + 2 + \varepsilon\}$ and for $T \ge R + 2$ there are positive constants $C(R,T)$ and A, which are independent of ε, such that

$$|\partial_t^k \partial_x^\beta a_{j,\alpha}(t,x;R,\varepsilon)| \le C(R,T) A_T^k (A/\varepsilon)^{|\alpha|} k! (\beta!)^{3/2}$$

for $\varepsilon \in (0,1]$, $k \in \mathbf{Z}_+$, $\beta \in (\mathbf{Z}_+)^n$, $t \in [-3\delta_1/2, T]$ and $x \in \mathbf{R}^n$, where $1 \le j \le m$, $|\alpha| = j-1$ and A_T is the constant in (A-2). We also choose $\rho^1(t) \in \mathcal{E}^{\{3/2\}}(\mathbf{R})$ so that $\operatorname{supp} \rho^1 \subset \{t \in \mathbf{R};\ 0 \le t \le 1\}$, $\rho^1(t) \ge 0$ and $\int_{-\infty}^{\infty} \rho^1(t)\, dt = 1$. Put $\rho^1_\varepsilon(t) = \varepsilon^{-1}\rho^1(\varepsilon^{-1}t)$. When $2 \le j \le m$ and $|\alpha| \le j-2$, we extend $a_{j,\alpha}(t,x)$ for $t \le 0$ as $a_{j,\alpha}(t,x) \in C^\infty(\mathbf{R}^{n+1})$, and define

$$a_{j,\alpha}(t,x;R,\varepsilon) = \Theta_{\delta_1}(-t) \int_{\mathbf{R}^{n+1}} \rho^1_\varepsilon(-t+s)\rho_\varepsilon(x-y)\Theta(|y|-R)a_{j,\alpha}(s,y)\, dsdy$$

for $(t,x) \in \mathbf{R}^{n+1}$. Then we have $a_{j,\alpha}(t,x;R,\varepsilon) \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ and $\operatorname{supp} a_{j,\alpha}(t,\cdot;R,\varepsilon) \subset \{x \in \mathbf{R}^n;\ |x| \le R+2+\varepsilon\}$ if $2 \le j \le m$ and $|\alpha| \le j-2$. Put

$$\begin{aligned}
P_m(t,x,\tau,\xi;R,\varepsilon) &= \hat{p}(t,\tau,\xi) \equiv \prod_{k=1}^m (\tau - \Theta_{\delta_1}(-t)\lambda_k(t,\xi)), \\
P_{m-1}(t,x,\tau,\xi;R,\varepsilon) &= \Theta_{\delta_1}(t) \sum_{j=1}^m \sum_{|\alpha|=j-1} a_{j,\alpha}(t,x;R,\varepsilon)\tau^{m-j}\xi^\alpha \\
&\quad + \frac{i}{2}(\Theta_{\delta_1}(t)\Theta_{\delta_1}(-t)\chi_{R,\varepsilon}(x) - 1)\partial_t\partial_\tau \hat{p}(t,\tau,\xi), \\
P_k(t,x,\tau,\xi;R,\varepsilon) &= \Theta_{\delta_1}(t) \sum_{j=m-k}^m \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t,x;R,\varepsilon)\tau^{m-j}\xi^\alpha \\
&\qquad (0 \le k \le m-2), \\
P(t,x,\tau,\xi;R,\varepsilon) &= \sum_{j=0}^m P_{m-j}(t,x,\tau,\xi;R,\varepsilon),
\end{aligned}$$

where $\chi_{R,\varepsilon}(x) = \int_{\mathbf{R}^n} \rho_\varepsilon(x-y)\Theta(|y|-R)\, dy$. It is easy to see that, for any $k \in \mathbf{Z}_+$ and $\beta \in (\mathbf{Z}_+)^n$,

$$\partial_t^k \partial_x^\beta a_{j,\alpha}(t,x;R,\varepsilon) \to \partial_t^k \partial_x^\beta (\Theta_{\delta_1}(-t)\Theta(|x|-R)a_{j,\alpha}(t,x)) \quad \text{uniformly in } (-\infty, 2\delta_1] \times \mathbf{R}^n \text{ as } \varepsilon \downarrow 0, \tag{2.11}$$

$$\partial_x^\beta \chi_{R,\varepsilon}(x) \to \partial_x^\beta \Theta(|x|-R) \quad \text{uniformly in } \mathbf{R}^n \text{ as } \varepsilon \downarrow 0, \tag{2.12}$$

where $1 \le j \le m$ and $|\alpha| < j$. We also put

$$\begin{aligned}
P(t,x,\tau,\xi;R) &= \hat{p}(t,\tau,\xi) + \sum_{j=1}^m P_{m-j}(t,x,\tau,\xi;R), \\
P_{m-1}(t,x,\tau,\xi;R) &= \Theta_{\delta_1}(t)\Theta_{\delta_1}(-t)\Theta(|x|-R)P_{m-1}(t,x,\tau,\xi) \\
&\quad + \frac{i}{2}(\Theta_{\delta_1}(t)\Theta(|x|-R) - 1)\partial_t\partial_\tau \hat{p}(t,\tau,\xi)
\end{aligned}$$

$$P_k(t,x,\tau,\xi;R) = \Theta_{\delta_1}(t)\Theta_{\delta_1}(-t)\Theta(|x|-R)P_k(t,x,\tau,\xi) \quad (\ 0 \le k \le m-2).$$

Note that

$$P(t,x,\tau,\xi;R) = \begin{cases} P(t,x,\tau,\xi) \\ \quad \text{if } 0 \le t \le 3\delta_1/2 \text{ and } |x| \le R+3/2, \\ p(t,\tau,\xi) - \dfrac{i}{2}\partial_t\partial_\tau p(t,\tau,\xi) \\ \quad \text{if } t \ge 2\delta_1 \text{ or ``} t \ge -3\delta_1/2 \text{ and } |x| \ge R+2\text{''}, \end{cases}$$

$$P(t,x,\tau,\xi;R,\varepsilon) = p(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p(t,\tau,\xi)$$
$$\text{if } t \ge 2\delta_1 \text{ or ``} t \ge -3\delta_1/2 \text{ and } |x| \ge R+2+\varepsilon\text{''},$$

$$(2.13) \quad sub\ \sigma(P(\cdot;R,\varepsilon))(t,x,\tau,\xi)$$
$$= \Theta_{\delta_1}(t)\int_{\mathbf{R}}^{n} \rho_\varepsilon(x-y)\Theta(|y|-R) sub\ \sigma(P)(t,y,\tau,\xi)\,dy$$
$$\text{if } t \ge -3\delta_1/2.$$

We factorized $p(t,\tau,\xi)$ as (2.9). By the factorization theorem we can write

$$(2.14) \quad P(t,x,\tau,\xi;R,\varepsilon) = P_{j,1}(t,x,\tau,\xi;R,\varepsilon) \circ P_{j,2}(t,x,\tau,\xi;R,\varepsilon)\circ$$
$$\cdots \circ P_{j,r_j+1}(t,x,\tau,\xi;R,\varepsilon) + R_j(t,x,\tau,\xi;R,\varepsilon)$$

for $1 \le j \le N_0$, $(t,x,\tau,\xi) \in [-3\delta_1/2, 4\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times (\overline{\mathcal{C}}_j \setminus \{0\})$ and $\varepsilon \in (0,1]$, where $P_{j,k}(t,x,\tau,\xi;R,\varepsilon) \in \mathcal{S}^2_{1,0}$ uniformly in ε ($1 \le k \le r_j$) and $P_{j,r_j+1}(t,x,\tau,\xi;R,\varepsilon) \in \mathcal{S}^{m-2r_j}_{1,0}$ uniformly in ε, the principal symbol of $P_{j,k}(t,x,\tau,\xi;R,\varepsilon)$ is equal to $p_{j,k}(t,\tau,\xi)$ and $R_j(t,x,\tau,\xi;R,\varepsilon) \in \mathcal{S}^{m-1,-\infty}_{1,0}$ uniformly in ε (see, *e.g.*, [8]). Here we denote by $a(t,x,\tau,\xi) \circ b(t,x,\tau,\xi)$ the symbol of $a(t,x,D_t,D_x)b(t,x,D_t,D_x)$. Since the $P_{j,k}(t,x,\tau,\xi;R,\varepsilon)$ are given by using contour integrals in $\mathbf{C}$, "uniformly in ε" follows from this construction. Indeed, let $m_1, m_2 \in \mathbf{N}$, and let $p_j(\tau) = \prod_{k=1}^{m_j}(\tau - \lambda_{j,k})$ ($j = 1,2$) be polynomials of τ. Assume that

$$\bigcap_{j=1}^{2}\{\lambda_{j,k};\ 1 \le k \le m_j\} = \emptyset.$$

In the construction, for a given polynomial $f(\tau)$ of degree m_1+m_2-1 we must find polynomials $g_j(\tau)$ ($j=1,2$) of degree m_j-1 satisfying $p_1(\tau)g_2(\tau)+p_2(\tau)g_1(\tau) = f(\tau)$. Choose rectifiable simple closed curves γ_j ($j=1,2$) in $\mathbf{C}$ so that $\{\lambda_{j,k};\ 1 \le k \le m_j\} \subset (\gamma_j)$ and the $\lambda_{j,k}$ do not belong to $\overline{(\gamma_{j+1})}$

($j = 1, 2$), where (γ_j) denotes the domain enclosed by γ_j and $\gamma_3 = \gamma_1$. Then $g_j(\tau)$ ($j = 1, 2$) are given by

$$g_j(\tau) = (2\pi i)^{-1} \oint_{\gamma_j} (p_j(\tau) - p_j(\lambda)) f(\lambda)((\tau - \lambda) p_1(\lambda) p_2(\lambda))^{-1} \, d\lambda$$

for $\tau \in \mathbf{C}$ (see Lemma 5.10 of [6]). Let us consider the relation between $sub\ \sigma(P(\cdot; R, \varepsilon))(t, x, \tau, \xi)$ and $sub\ \sigma(P_{j,k}(\cdot; R, \varepsilon))(t, x, \tau, \xi)$. Write

$$\begin{aligned} P_{j,k}(t, x, \tau, \xi; R, \varepsilon) &= \tilde{p}_{j,k}(t, \tau, \xi) + q_{j,k}(t, x, \tau, \xi; R, \varepsilon), \\ q_{j,k}(t, x, \tau, \xi; R, \varepsilon) &= q^1_{j,k,0}(t, x, \xi; R, \varepsilon)\tau + q^1_{j,k,1}(t, x, \xi; R, \varepsilon) \\ &\quad + q^0_{j,k}(t, x, \tau, \xi; R, \varepsilon) \end{aligned}$$

for $1 \le j \le N_0$, $1 \le k \le r_j$ and $t \ge -3\delta_1/2$, where $q^1_{j,k,l}(t, x, \xi; R, \varepsilon) \in S^l_{1,0}(\mathbf{R} \times T^*\mathbf{R}^n)$ uniformly in ε ($l = 0, 1$), $q^0_{j,k}(t, x, \tau, \xi; R, \varepsilon) \in \mathcal{S}^{1,-1}_{1,0}$ uniformly in ε and $q^1_{j,k,l}(t, x, \xi; R, \varepsilon)$ ($l = 0, 1$) are positively homogeneous of degree l for $|\xi| \ge 1/4$. We also write

$$P_{j,r_j+1}(t, x, \tau, \xi; R, \varepsilon) = \tilde{p}_{j,r_j+1}(t, \tau, \xi) + q_{j,r_j+1}(t, x, \tau, \xi; R, \varepsilon)$$

for $1 \le j \le N_0$ and $t \ge -3\delta_1/2$, where $q_{j,r_j+1}(t, x, \tau, \xi; R, \varepsilon) \in \mathcal{S}^{m-2r_j-1}_{1,0}$ uniformly in ε.

Lemma 2.3. *For* $1 \le j \le N_0$ *and* $(t, x, \tau, \xi) \in [0, 3\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times (\overline{\mathcal{C}}_j \cap S^{n-1})$ *we have*

$$\begin{aligned} &sub\ \sigma(P(\cdot; R, \varepsilon))(t, x, \tau, \xi) \\ &= \sum_{k=1}^{r_j+1} sub\ \sigma(P_{j,k}(\cdot; R, \varepsilon))(t, x, \tau, \xi) \prod_{1 \le l \le r_j+1,\, l \ne k} p_{j,l}(t, \tau, \xi) \\ &\quad + \frac{i}{2} \sum_{1 \le k < l \le r_j+1} \{p_{j,l}(t, \tau, \xi), p_{j,k}(t, \tau, \xi)\} \prod_{1 \le \mu \le r_j+1,\, \mu \ne k,l} p_{j,\mu}(t, \tau, \xi), \end{aligned}$$

where $\{a(t, \tau, \xi), b(t, \tau, \xi)\} = \partial_\tau a(t, \tau, \xi) \cdot \partial_t b(t, \tau, \xi) - \partial_t a(t, \tau, \xi) \cdot \partial_\tau b(t, \tau, \xi)$.

Proof. We can prove by induction on r that

$$\begin{aligned} (2.15) \qquad & P_1(t, x, \tau, \xi) \circ P_2(t, x, \tau, \xi) \circ \cdots \circ P_r(t, x, \tau, \xi) \\ & - \Big\{ \prod_{k=1}^{r} \tilde{p}_k(t, \tau, \xi) + \sum_{k=1}^{r} q^0_k(t, x, \tau, \xi) \prod_{1 \le l \le r,\, l \ne k} \tilde{p}_l(t, \tau, \xi) \\ & - \sum_{1 \le k < l \le r} i\partial_\tau \tilde{p}_k(t, \tau, \xi) \cdot \partial_t \tilde{p}_l(t, \tau, \xi) \cdot \prod_{1 \le \mu \le r,\, \mu \ne k,l} \tilde{p}_\mu(t, \tau, \xi) \Big\} \end{aligned}$$

$$\in \mathcal{S}_{1,0}^{m-1,-1},$$

where $m_1, \cdots, m_r \in \mathbf{N}$, $m = m_1 + \cdots + m_r$, $P_k(t,x,\tau,\xi) = \tilde{p}_k(t,\tau,\xi) + q_k(t,x,\tau,\xi) \in \mathcal{S}_{1,0}^{m_k}$, $\tilde{p}_k(t,\tau,\xi)$ and $q_k^0(t,x,\tau,\xi)$ coincide with the principal symbols of $P_k(t,x,\tau,\xi)$ and $q_k(t,x,\tau,\xi)$ for $|\xi| \ge 1/4$, respectively, and $\prod_{1\le\mu\le r,\,\mu\ne k,l} \cdots = 1$ if $r = 2$. Since

$$\begin{aligned}\frac{i}{2}\partial_t\partial_\tau \prod_{k=1}^r \tilde{p}_k(t,\tau,\xi) &= \frac{i}{2}\sum_{k=1}^r \partial_t\partial_\tau \tilde{p}_k(t,\tau,\xi)\cdot \prod_{1\le l\le r,\, l\ne k} \tilde{p}_l(t,\tau,\xi)\\ &+ \frac{i}{2}\sum_{1\le k<l\le r} (\partial_\tau \tilde{p}_k(t,\tau,\xi)\cdot\partial_t\tilde{p}_l(t,\tau,\xi) + \partial_t\tilde{p}_k(t,\tau,\xi)\cdot\partial_\tau\tilde{p}_l(t,\tau,\xi))\\ &\times \prod_{1\le\mu\le r,\,\mu\ne k.l} \tilde{p}_\mu(t,\tau,\xi),\end{aligned}$$

(2.15) proves the lemma. □

Let $1 \le j \le N_0$, and write

$$p_{j,k}(t,\tau,\xi) = \prod_{\mu=1}^2 (\tau - \lambda_{j,k,\mu}(t,\xi)) \quad (\,1 \le k \le r_j),$$
$$p_{j,r_j+1}(t,\tau,\xi) = \prod_{\mu=1}^{m-2r_j} (\tau - \lambda_{j,r_j+1,\mu}(t,\xi)),$$

and put

$$p_{j,k,\mu}(t,\tau,\xi) = \tau - \lambda_{j,k,\nu}(t,\xi) \quad \text{for } 1\le k\le r_j \text{ if } \{\mu,\nu\} = \{1,2\}, \tag{2.16}$$
$$p_{j,r_j+1,\mu}(t,\tau,\xi) = \prod_{1\le\nu\le m-2r_j,\,\nu\ne\mu} (\tau - \lambda_{j,r_j+1,\nu}(t,\xi)) \quad \text{for } 1 \le \mu \le m-2r_j. \tag{2.17}$$

Moreover, we put

$$\begin{aligned}d_0 = \min\{&|\lambda_{j,k,\mu}(t,\xi) - \lambda_{j,l,\nu}(t,\xi)|;\ 1\le k<l\le r_j+1,\ 1\le\mu\le m_{j,k},\\ &1\le\nu\le m_{j,l},\ t\in[0,3\delta_1] \text{ and } \xi \in \overline{\mathcal{C}}_j \cap S^{n-1}\},\end{aligned}$$

where $m_{j,k} = 2$ if $1 \le k \le r_j$ and $m_{j,r_j+1} = m - 2r_j$. From (2.10) we have $d_0 > 0$. Let $1 \le k \le r_j$. It follows from Lemma 2.3 that

$$\mathit{sub}\ \sigma(P_{j,k}(\cdot;R,\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \prod_{1\le l\le r_j+1,\, l\ne k} p_{j,l}(t,b_{j,k}(t,\xi),\xi) \tag{2.18}$$

$$
\begin{aligned}
&= sub\ \sigma(P(\cdot; R,\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \\
&\quad + \sum_{1\le l\le r_j+1,\, l\ne k} sub\ \sigma(P_{j,l}(\cdot;R,\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \\
&\qquad\qquad \times a_{j,k}(t,\xi) \prod_{1\le \mu\le r_j+1,\, \mu\ne k,l} p_{j,\mu}(t,b_{j,k}(t,\xi),\xi) \\
&\quad + \frac{i}{2}\Big\{\sum_{l=1}^{k-1} - \sum_{l=k+1}^{r_j+1}\Big\}\{p_{j,l}(t,\tau,\xi), p_{j,k}(t,\tau,\xi)\}|_{\tau=b_{j,k}(t,\xi)} \\
&\qquad\quad \times \prod_{1\le \mu\le r_j+1,\, \mu\ne k,l} p_{j,\mu}(t,b_{j,k}(t,\xi),\xi) \\
&\quad + \frac{i}{2} \sum_{1\le \mu<\nu\le r_j+1,\, \mu\ne k,\, \nu\ne k} \{p_{j,\nu}(t,\tau,\xi), p_{j,\mu}(t,\tau,\xi)\}|_{\tau=b_{j,k}(t,\xi)} \\
&\qquad\qquad \times a_{j,k}(t,\xi) \prod_{1\le s\le r_j+1,\, s\ne k,\mu,\nu} p_{j,s}(t,b_{j,k}(t,\xi),\xi)
\end{aligned}
$$

for $t \in [0, 3\delta_1]$. Note that

$$\partial_\tau p_{j,k}(t,\tau,\xi)|_{\tau=b_{j,k}(t,\xi)} = 0, \tag{2.19}$$

$$\partial_t p_{j,k}(t,\tau,\xi)|_{\tau=b_{j,k}(t,\xi)} = -\partial_t a_{j,k}(t,\xi). \tag{2.20}$$

We may assume that $d_k \equiv \inf\{|b_{j,k}(t,\xi) - \lambda_{j,l,\mu}(t,\xi)|;\ 1 \le l \le r_j + 1$ with $l \ne k$, $1 \le \mu \le m_{j,l}$ and $(t,\xi) \in [0,3\delta_1] \times (\overline{\mathcal{C}}_j \cap S^{n-1})\} > 0$, modifying $\mathcal{C}_j$ if necessary. Put

$$\tilde{d}_0 = \min\{d_0, d_1, \cdots, d_{r_j}\}.$$

Then we have

$$\Big| \prod_{1\le l\le r_j+1,\, l\ne k} p_{j,l}(t,b_{j,k}(t,\xi),\xi)\Big|^{-1} \le \tilde{d}_0^{-m+2}|\xi|^{-m+2} \tag{2.21}$$

for $t \in [0,3\delta_1]$ and $\xi \in (\overline{\mathcal{C}}_j \setminus \{0\})$. From (2.18) – (2.21) we have the following

Lemma 2.4. *There are symbols* $c_{j,k,0}(t,x,\xi), c_{j,k,1}(t,\xi) \in S^{-1}_{1,0}(\mathbf{R}\times T^*\mathbf{R}^n)$ ($1 \le j \le N_0$, $1 \le k \le r_j$) *such that*

$$
\begin{aligned}
&sub\ \sigma(P_{j,k}(\cdot;R,\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \\
&= sub\ \sigma(P(\cdot;R,\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) / \prod_{1\le l\le r_j+1,\, l\ne k} p_{j,l}(t,b_{j,k}(t,\xi),\xi) \\
&\quad + c_{j,k,0}(t,x,\xi)a_{j,k}(t,\xi) + c_{j,k,1}(t,\xi)\partial_t a_{j,k}(t,\xi)
\end{aligned}
$$

for $1 \le j \le N_0$, $1 \le k \le r_j$ *and* $(t,\tau,\xi) \in [0,3\delta_1] \times \mathbf{R}^n \times \mathcal{C}_j$ *with* $|\xi| \ge 1$.

Let $\mathcal{O}_0$ be the ring of (convergent) power series centered at $t=0$ in one variable. We note that $\mathcal{O}_0$ is a principal ideal ring. Define

$$\begin{aligned}\mathfrak{M}_0 =&\{(\beta_{j,\alpha}(t))_{j+|\alpha|=m-1} \in \mathcal{O}_0^{M'};\ \text{there are } C>0 \text{ and } \delta>0 \text{ such that}\\ &\quad \min\{\min_{s\in\mathcal{R}(\xi)} |t-s|,1\}\Big|\sum_{j+|\alpha|=m-1} \beta_{j,\alpha}(t)\tau^j\xi^\alpha\Big| \le Ch_{m-1}(t,\tau,\xi)^{1/2}\\ &\quad \text{for } t\in[0,\delta],\ \tau\in\mathbf{R} \text{ and } \xi\in S^{n-1}\},\end{aligned}$$

where $M' = \binom{m+n-1}{m-1}$. Note that each $\mathcal{O}_0$-submodule of $\mathcal{O}_0^{M'}$ is finitely generated (see, *e.g.*, §6.3 of [5] and [3]). Therefore, there are $r_0 \in \mathbf{N}$ and $\beta^\mu(t) \equiv (\beta^\mu_{j,\alpha}(t))_{j+|\alpha|=m-1} \in \mathfrak{M}_0$ ($1\le\mu\le r_0$) such that

$$\mathfrak{M}_0 = \Big\{\sum_{\mu=1}^{r_0} c_\mu(t)\beta^\mu(t);\ c_\mu(t)\in\mathcal{O}_0\ (\ 1\le\mu\le r_0)\Big\}.$$

By the assumption (L) there are $c_\mu(t,x)\in C^\infty([0,3\delta_1]\times\mathbf{R}^n)$ ($1\le\mu\le r_0$) such that

$$\textit{sub}\ \sigma(P)(t,x,\tau,\xi) = \sum_{\mu=1}^{r_0} c_\mu(t,x)\beta^\mu(t,\tau,\xi), \tag{2.22}$$

where $\beta^\mu(t,\tau,\xi) = \sum_{j+|\alpha|=m-1} \beta^\mu_{j,\alpha}(t)\tau^j\xi^\alpha$, modifying δ_1 if necessary. Here $c_\mu(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$ ($1\le\mu\le r_0$) follows from the construction of the $\beta^\mu(t,\tau,\xi)$ (see [3] and the proof of Lemma 3.1 of [12]). Moreover, we may assume that

$$\min\{\min_{s\in\mathcal{R}(\xi)} |t-s|,1\}|\beta^\mu(t,\tau,\xi)| \le Ch_{m-1}(t,\tau,\xi)^{1/2}$$

for $(t,\tau,\xi)\in[0,3\delta_1]\times\mathbf{R}\times S^{n-1}$. Instead of the Cauchy problem (CP) we consider

$$\text{(CP)}'\qquad \begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty)\times\mathbf{R}^n,\\ D_t^j u(t,x)|_{t=0} = 0 & \text{in } \mathbf{R}^n\ (\ j\in\mathbf{Z}_+),\end{cases}$$

where $f(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfies $D_t^j f(t,x)|_{t=0}=0$ ($j\in\mathbf{Z}_+$). It is easy to see that (CP) is also solvable in $C^\infty([0,\infty)\times\mathbf{R}^n)$ if (CP)′ is solvable in $C^\infty([0,\infty)\times\mathbf{R}^n)$ for any $f(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$ with $D_t^j f(t,x)|_{t=0}=0$ ($j\in\mathbf{Z}_+$). Let $f(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfy $D_t^j f(t,x)|_{t=0}=0$ ($j\in\mathbf{Z}_+$), and let $R\ge1$ and $\varepsilon\in(0,1]$. Put $\tilde f(t,x) = \begin{cases} f(t,x) & (\ t\ge0),\\ 0 & (\ t<0).\end{cases}$ We define

$$f_{R,\varepsilon}(t,x) = \Theta_{2\delta_1}(t)\int_{\mathbf{R}^{n+1}} \rho^1_\varepsilon(t-s)\rho_\varepsilon(x-y)\Theta(|y|-R)\tilde f(s,y)\,dsdy. \tag{2.23}$$

Then we have $f_{R,\varepsilon} \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ and

$$\operatorname{supp} f_{R,\varepsilon} \subset \{(t,x) \in \mathbf{R}^{n+1};\ 0 \le t \le 4\delta_1 \text{ and } |x| \le R+2+\varepsilon\}.$$

Moreover, we have

$$f_{R,\varepsilon}(t,x) \to \Theta_{2\delta_1}(t)\Theta(|x|-R)\tilde{f}(t,x) \text{ in } C_0^\infty(\mathbf{R}^{n+1}) \text{ as } \varepsilon \downarrow 0. \tag{2.24}$$

To construct solutions to (CP)$'$ we first consider

$$(\mathrm{CP})_{R,\varepsilon} \qquad \begin{cases} P(t,x,D_t,D_x;R,\varepsilon)v_{R,\varepsilon}(t,x) = f_{R,\varepsilon}(t,x), \\ \operatorname{supp} v_{R,\varepsilon} \subset [0,\infty) \times \mathbf{R}^n. \end{cases}$$

It is well-known that $(\mathrm{CP})_{R,\varepsilon}$ has a unique solution $v_{R,\varepsilon}$ in $\mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$, and that $(t_0,x^0) \notin \operatorname{supp} v_{R,\varepsilon}$ if $(t_0,x^0) \in (0,\infty) \times \mathbf{R}^n$ and $f_{R,\varepsilon}(t,x) = 0$ near $K^-_{(t_0,x^0)}$ $(\cap[0,\infty) \times \mathbf{R}^n)$ (see, *e.g.*, [9]). We shall derive energy estimates for $P(t,x,D_t,D_x;R,\varepsilon)$. Let $v(t,x) \in C^\infty(\mathbf{R};H^\infty(\mathbf{R}^n_x))$ satisfy $v|_{t\le 0} = 0$, and put

$$g_{R,\varepsilon} = P(t,x,D_t,D_x;R,\varepsilon)v.$$

Here $H^s(\mathbf{R}^n)$ denotes the Sobolev space of order s and $H^\infty(\mathbf{R}^n) = \bigcap_{s\in\mathbf{R}} H^s(\mathbf{R}^n)$. Note that

$$P(t,x,D_t - i\gamma,D_x;R,\varepsilon)(e^{-\gamma t}v) = e^{-\gamma t}g_{R,\varepsilon},$$

where $\gamma \ge 1$. Let $\chi_j(t) \in C^\infty(\mathbf{R})$ ($j=0,1$) satisfy

$$\chi_0(t) = \begin{cases} 1 & \text{if } -\delta_1 \le t \le 3\delta_1, \\ 0 & \text{if } t \le -2\delta_1 \text{ or } t \ge 4\delta_1, \end{cases}$$

$$\chi_1(t) = \begin{cases} 1 & \text{if } t \le 4\delta_1, \\ 0 & \text{if } t \ge 5\delta_1. \end{cases}$$

Then we have

$$\begin{aligned} &P(t,x,D_t - i\gamma,D_x;R,\varepsilon)(e^{-\gamma t}\chi_1(t)v) \\ &\quad = e^{-\gamma t}\chi_1(t)g_{R,\varepsilon} + [P(t,x,D_t - i\gamma,D_x;R,\varepsilon),\chi_1(t)](e^{-\gamma t}v), \end{aligned} \tag{2.25}$$

where $[A,B] = AB - BA$ for operators A and B. Let us estimate $\Theta_\gamma(D_x)(e^{-\gamma t} \times \chi_0(t)v)$. Put

$$C_0 = \max\{4|\lambda_j(t,\xi)|;\ t \in [-2\delta_1,4\delta_1],\ \xi \in S^{n-1} \text{ and } 1 \le j \le m\}.$$

Suppose that $t \in [-2\delta_1, 4\delta_1]$ and $|\xi| \le 2\gamma$. If $|\tau| \le C_0\gamma$, then there is $C_1 > 0$, which is independent of γ and $\varepsilon \in (0,1]$, satisfying

$$|P(t,x,\tau - i\gamma,\xi;R,\varepsilon)| \ge |\hat{p}(t,\tau - i\gamma,\xi)| - \sum_{j=1}^{m} |P_{m-j}(t,x,\tau - i\gamma,\xi;R,\varepsilon)|$$
$$\ge \gamma^m - C_1\gamma^{m-1}.$$

Hereafter the constants do not depend on $\gamma \ge 1$ and $\varepsilon \in (0,1]$ unless stated. Therefore, we have

$$|P(t,x,\tau - i\gamma,\xi;R,\varepsilon)| \ge \gamma^m/2 \quad \text{if } |\tau| \le C_0\gamma \text{ and } \gamma \ge 2C_1.$$

Moreover, there is $C_2 > 0$ satisfying

$$|P(t,x,\tau - i\gamma,\xi;R,\varepsilon)| \ge 2^{-m}(\tau^2+\gamma^2)^{m/2} - C_2(\tau^2+\gamma^2)^{(m-1)/2}$$
$$\ge 2^{-m-1}(\tau^2+\gamma^2)^{m/2}$$

if $|\tau| \ge C_0\gamma$ and $\gamma \ge 2^{m+1}C_2$. Therefore, there is $c_0 > 0$ such that

$$|P(t,x,\tau - i\gamma,\xi;R,\varepsilon)| \ge c_0\langle(\tau,\xi)\rangle_\gamma^m \quad \text{for } (t,x,\tau,\xi) \in [-2\delta_1,4\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \text{ with } |\xi| \le 2\gamma, \tag{2.26}$$

where $\langle(\tau,\xi)\rangle_\gamma = (\gamma^2+\tau^2+|\xi|^2)^{1/2}$. (2.26) implies that $P(t,x,\tau-i\gamma,\xi;R,\varepsilon)$ is elliptic in $\{(t,x,\tau,\xi) \in [-2\delta_1,4\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n;\ |\xi| \le 2\gamma\}$. It is obvious that, with some positive constants $C_{j,k,\alpha,\beta}$ and C_α,

$$|D_t^k D_x^\beta \partial_\tau^j \partial_\xi^\alpha P(t,x,\tau-i\gamma,\xi;R,\varepsilon)^{-1}| \le C_{j,k,\alpha,\beta}\langle(\tau,\xi)\rangle_\gamma^{-m-j-|\alpha|}$$
$$\text{for } (t,x,\tau,\xi) \in [-2\delta_1,4\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \text{ with } |\xi| \le 2\gamma,$$
$$|\partial_\xi^\alpha \Theta_\gamma(\xi)| \le C_\alpha\langle\xi\rangle_\gamma^{-|\alpha|},$$

where $\langle\xi\rangle_\gamma = (\gamma^2+|\xi|^2)^{1/2}$. Define, inductively,

$$E_0(t,x,\tau,\xi;\gamma;R,\varepsilon) = \chi_0(t)\Theta_\gamma(\xi)P(t,x,\tau-i\gamma,\xi;R,\varepsilon)^{-1},$$
$$E_k(t,x,\tau,\xi;\gamma;R,\varepsilon)$$
$$= -\sum_{\substack{\tilde\alpha\in(\mathbf{Z}_+)^{n+1},\,|\tilde\alpha|+\mu=k\\ 0\le\mu\le k-1}} \frac{1}{\tilde\alpha!}E_\mu^{(\tilde\alpha)}(t,x,\tau,\xi;\gamma;R,\varepsilon)P_{(\tilde\alpha)}(t,x,\tau-i\gamma,\xi;R,\varepsilon)$$
$$\times P(t,x,\tau-i\gamma,\xi;R,\varepsilon)^{-1} \quad (\ k=1,2,\cdots),$$

where $f_{(\tilde\beta)}^{(\tilde\alpha)}(t,x,\tau,\xi) = D_t^l D_x^\beta \partial_\tau^j \partial_\xi^\alpha f(t,x,\tau,\xi)$ for $\tilde\alpha = (j,\alpha) \in (\mathbf{Z}_+)^{n+1}$ and $\tilde\beta = (l,\beta) \in (\mathbf{Z}_+)^{n+1}$. Then it is easy to see that, with $C_{k,\tilde\alpha,\tilde\beta} > 0$,

$$|E_{k(\tilde\beta)}^{(\tilde\alpha)}(t,x,\tau,\xi;\gamma;R,\varepsilon)| \le C_{k,\tilde\alpha,\tilde\beta}\langle(\tau,\xi)\rangle_\gamma^{-m-j}\langle\xi\rangle_\gamma^{-k-|\alpha|} \tag{2.27}$$

for $k \in \mathbf{Z}_+$, $\tilde{\alpha} = (j,\alpha) \in (\mathbf{Z}_+)^{n+1}$, $\tilde{\beta} = (l,\beta) \in (\mathbf{Z}_+)^{n+1}$ and $(t,x,\tau,\xi) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. Define a Riemannian metric g_0 in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ by

$$g_{0\,(t,x,\tau,\xi)} = (dt)^2 + |dx|^2 + \langle(\tau,\xi)\rangle_\gamma^{-2}(d\tau)^2 + \langle\xi\rangle_\gamma^{-2}|d\xi|^2.$$

We can easily prove that g_0 is uniformly σ temperate in γ. Here we refer to [4] for the definition of σ temperate. And "uniformly in $\gamma, \cdots$" implies that the constants appearing in the definition do not depend on $\gamma, \cdots$. Moreover, $\langle(\tau,\xi)\rangle_\gamma$ and $\langle\xi\rangle_\gamma$ are uniformly σ,g_0 temperate in γ. (2.27) gives

$$E_k(t,x,\tau,\xi;\gamma;R,\varepsilon) \in S(\langle(\tau,\xi)\rangle_\gamma^{-m}\langle\xi\rangle_\gamma^{-k}, g_0) \quad \text{uniformly in } \gamma \text{ and } \varepsilon.$$

Here we also refer to [4] for the terminologies and notations. For $N \in \mathbf{Z}_+$ we put

$$E^N(t,x,\tau,\xi;\gamma;R,\varepsilon) = \sum_{k=0}^{N} E_k(t,x,\tau,\xi;\gamma;R,\varepsilon)$$
$$(\in S(\langle(\tau,\xi)\rangle_\gamma^{-m}, g_0) \text{ uniformly in } \gamma \text{ and } \varepsilon).$$

Then we have

$$\text{(2.28)} \qquad E^N(t,x,\tau,\xi;\gamma;R,\varepsilon) \circ P(t,x,\tau - i\gamma,\xi;R,\varepsilon) - \chi_0(t)\Theta_\gamma(\xi)$$
$$\in S(\langle\xi\rangle_\gamma^{-N-1}, g_0) \quad \text{uniformly in } \gamma \text{ and } \varepsilon.$$

Since $E^N(t,x,\tau,\xi;\gamma;R,\varepsilon) = 0$ if $d\chi_1(t) \neq 0$, we have

$$\text{(2.29)} \qquad \sigma(E^N(t,x,D_t,D_x;\gamma;R,\varepsilon)[P(t,x,D_t - i\gamma,D_x;R,\varepsilon),\chi_1(t)])$$
$$\in S(\langle(\tau,\xi)\rangle_\gamma^{-1}\langle\xi\rangle_\gamma^{-k}, g_0) \quad \text{uniformly in } \gamma \text{ and } \varepsilon$$

for any $k \in \mathbf{Z}_+$, where $\sigma(a(t,x,D_t,D_x)) = a(t,x,\tau,\xi)$. Let $\chi_2(t) \in C_0^\infty(\mathbf{R})$ satisfy

$$\chi_2(t) = \begin{cases} 1 & \text{if } 7\delta_1/2 \le t \le 11\delta_1/2, \\ 0 & \text{if } t \le 3\delta_1 \text{ or } t \ge 6\delta_1. \end{cases}$$

Then we have

$$[P(t,x,D_t - i\gamma,D_x;R,\varepsilon),\chi_1(t)]v = [P(\cdots),\chi_1](\chi_2(t)v).$$

Multiplying (2.25) by $\langle(D_t,D_x)\rangle_\gamma^m\langle D_x\rangle_\gamma^l E^N(t,x,D_t,D_x;\gamma;R,\varepsilon)$, (2.28) and (2.29) give the following

Lemma 2.5. *For any $l \in \mathbf{R}$ and any $N \in \mathbf{N}$ there are positive constants C_l and $C_{l,N}$ such that C_l is independent of N and*

$$\begin{aligned}
&\|\langle(D_t, D_x)\rangle_\gamma^m \langle D_x\rangle_\gamma^l \Theta_\gamma(D_x)(e^{-\gamma t}\chi_0(t)v)\|_{L^2(\mathbf{R}^{n+1})} \\
&\le C_l \|\langle D_x\rangle_\gamma^l (e^{-\gamma t}\chi_1(t) g_{R,\varepsilon})\|_{L^2(\mathbf{R}^{n+1})} \\
&\quad + C_{l,N}\{\|\langle(D_t, D_x)\rangle_\gamma^m \langle D_x\rangle_\gamma^{-N}(e^{-\gamma t}\chi_1(t)v)\|_{L^2(\mathbf{R}^{n+1})} \\
&\qquad + \|\langle(D_t, D_x)\rangle_\gamma^{m-1}\langle D_x\rangle_\gamma^{-N}(e^{-\gamma t}\chi_2(t)v)\|_{L^2(\mathbf{R}^{n+1})}\}.
\end{aligned}$$

Choose $\chi_3(t) \in C^\infty(\mathbf{R})$ so that

$$\chi_3(t) = \begin{cases} 0 & \text{if } t \le 2\delta_1, \\ 1 & \text{if } t \ge 3\delta_1. \end{cases}$$

Then we have

$$P(t, x, D_t, D_x; R, \varepsilon)(\chi_3(t)v) = \chi_3(t) g_{R,\varepsilon} + [P_{R,\varepsilon}, \chi_3]v, \tag{2.30}$$

where $P_{R,\varepsilon} = P(t, x, D_t, D_x; R, \varepsilon)$. Since $\operatorname{supp}\chi_3 \subset [2\delta_1, \infty)$, (2.30) yields

$$\begin{aligned}
&\Big(p(t, D_t, D_x) - \frac{i}{2}Op(\partial_t\partial_\tau p(t,\tau,\xi))\Big)(\chi_3(t)v) \\
&\quad = \chi_3(t) g_{R,\varepsilon} + \Big[p(t, D_t, D_x) - \frac{i}{2}Op(\partial_t\partial_\tau p(t,\tau,\xi)), \chi_3\Big]v,
\end{aligned} \tag{2.31}$$

where $Op(a(t,x,\tau,\xi)) = a(t,x,D_t,D_x)$. Since $p(t, D_t, D_x) - (i/2)Op(\partial_t\partial_\tau p(t, \tau,\xi))$ has time dependent coefficients and $D_t^k(\chi_3(t)v(t,x)) = 0$ for $t \le 2\delta_1$ and $k \in \mathbf{Z}_+$, we can apply Lemma 3.2 of [13] to (2.31), replacing $\langle D_x\rangle$ by $\langle D_x\rangle_\gamma$. Therefore, there are $C > 0$ and $\nu_0 > 0$ such that

$$\begin{aligned}
&\sum_{k=0}^m \|\langle D_x\rangle_\gamma^{l+m-k} D_t^k(\chi_3(t)v(t,x))\|_{L^2(\mathbf{R}_x^n)}^2 \\
&\le C\Big\{\int_{2\delta_1}^t \|\langle D_x\rangle_\gamma^{l+\nu_0}\chi_3(s) g_{R,\varepsilon}(s,x)\|_{L^2(\mathbf{R}_x^n)}^2\, ds \\
&\qquad + \int_{2\delta_1}^t \|\langle D_x\rangle_\gamma^{l+\nu_0}[p - (i/2)Op(\partial_t\partial_\tau p), \chi_3]v(t,x)|_{t=s}\|_{L^2(\mathbf{R}_x^n)}^2\, ds\Big\}
\end{aligned} \tag{2.32}$$

for $l \in \mathbf{R}$ and $t \in [2\delta_1, 6\delta_1]$. Note that $\chi_3(t) = 1$ for $t \ge 3\delta_1$, $\operatorname{supp} d\chi_3 \subset [2\delta_1, 3\delta_1]$ and $e^{-2\gamma t} \le e^{-2\gamma s}$ for $s \in [2\delta_1, t]$. Multiplying (2.32) by $e^{-2\gamma t}$, we have

$$\sum_{k=0}^m \|e^{-\gamma t} D_t^k \langle D_x\rangle_\gamma^{l+m-k} v(t,x)\|_{L^2(\mathbf{R}_x^n)}^2$$

$$
\begin{aligned}
&\le C'\Big\{\int_{2\delta_1}^{t} \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_0} g_{R,\varepsilon}(s,x)\|^2_{L^2(\mathbf{R}^n_x)}\,ds \\
&\qquad + \sum_{k=0}^{m-1}\int_{2\delta_1}^{3\delta_1} \|e^{-\gamma s}D_t^k\langle D_x\rangle_\gamma^{l+\nu_0+m-k-1} v(t,x)|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\Big\}
\end{aligned}
$$

for $t \in [3\delta_1, 6\delta_1]$, where $C' > 0$. This gives the following

Lemma 2.6. *There is $C > 0$ satisfying*

$$
\begin{aligned}
&\sum_{k=0}^{m}\int_{3\delta_1}^{6\delta_1} \|e^{-\gamma t}D_t^k\langle D_x\rangle_\gamma^{l+m-k} v(t,x)\|^2_{L^2(\mathbf{R}^n_x)}\,dt \\
&\le C\Big\{\int_{2\delta_1}^{6\delta_1} \|e^{-\gamma t}\langle D_x\rangle_\gamma^{l+\nu_0} g_{R,\varepsilon}(t,x)\|^2_{L^2(\mathbf{R}^n_x)}\,dt \\
&\qquad + \sum_{k=0}^{m-1}\int_{2\delta_1}^{3\delta_1} \|e^{-\gamma t}D_t^k\langle D_x\rangle_\gamma^{l+\nu_0+m-k-1} v(t,x)\|^2_{L^2(\mathbf{R}^n_x)}\,dt\Big\}.
\end{aligned}
$$

Let $\mathcal{C}_{j,k}$ ($1 \le j \le N_0$, $1 \le k \le 4$) be open conic sets in $\mathbf{R}^n \setminus \{0\}$ satisfying $\mathcal{C}_{j,0} \Subset \mathcal{C}_{j,1} \Subset \mathcal{C}_{j,2} \Subset \mathcal{C}_{j,3} \Subset \mathcal{C}_{j,4} \Subset \mathcal{C}_j$. Choose $\Psi_j(\xi)$, $\varphi_j(\xi) \in S^0_{1,0}$ ($1 \le j \le N_0$) so that

$$
\Psi_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{C}_{j,1} \text{ and } |\xi| \ge 1, \\ 0 & \text{if } \xi \notin \mathcal{C}_{j,2} \text{ or } |\xi| \le 1/2, \end{cases}
$$

$$
\varphi_j(\xi) = \begin{cases} 0 & \text{if } \xi \in \mathcal{C}_{j,0} \text{ or } |\xi| \le 1/4, \\ 1 & \text{if } \xi \notin \mathcal{C}_{j,1} \text{ and } |\xi| \ge 1/2. \end{cases}
$$

Put $\Psi_{j,\gamma}(\xi) = (1 - \Theta_{3\gamma/4}(\xi))\Psi_j(\xi)$ for $\gamma \ge 1$. Then we have

$$
\begin{aligned}
&P(t,x,D_t,D_x;R,\varepsilon)\Psi_{j,\gamma}(D_x)v(t,x) \\
&\quad = \Psi_{j,\gamma}(D_x)g_{R,\varepsilon}(t,x) + [P_{R,\varepsilon},\Psi_{j,\gamma}]v(t,x) \quad (1 \le j \le N_0).
\end{aligned}
\tag{2.33}
$$

It is obvious that $[p(t,D_t,D_x),\Psi_{j,\gamma}(D_x)] = 0$, $\operatorname{supp}\sigma([P_{R,\varepsilon},\Psi_{j,\gamma}])(t,x,\tau,\xi) \subset [-2\delta_1, 2\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$, and there are $C_j(t,x,\tau,\xi;R,\varepsilon,\gamma) \in \mathcal{S}^{m-1}_{1,0}$ uniformly in γ and ε ($1 \le j \le N_0$) satisfying

$$
\begin{aligned}
&\sigma([P_{R,\varepsilon},\Psi_{j,\gamma}])(t,x,\tau,\xi) - C_j(t,x,\tau,\xi;R,\varepsilon,\gamma) \\
&\qquad\qquad \in \mathcal{S}^{m-1,-\infty}_{1,0} \text{ uniformly in } \gamma \text{ and } \varepsilon, \\
&\operatorname{supp} C_j(t,x,\tau,\xi;R,\varepsilon,\gamma) \\
&\subset \{(t,x,\tau,\xi) \in [-2\delta_1, 2\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times \mathcal{C}_{j,2};
\end{aligned}
$$

$$|x| \le R+3,\ |\xi| \ge 9\gamma/8 \text{ and "}\xi \notin \mathcal{C}_{j,1} \text{ or } |\xi| \le 3\gamma/2\text{"}\}.$$

We put

$$\Lambda_j(\xi) = \varphi_j(\xi)\log(1+\langle\xi\rangle) \quad (\ 1 \le j \le N_0).$$

For $B \ge 1$ we define

$$P_{B\Lambda_j}(t,x,\tau,\xi;R,\varepsilon) = e^{-B\Lambda_j(\xi)} \circ P(t,x,\tau,\xi;R,\varepsilon)e^{B\Lambda_j(\xi)}.$$

From (2.14) and (2.33) we have

$$\begin{aligned}(2.34)\qquad &(P_{j,1})_{B\Lambda_j}(P_{j,2})_{B\Lambda_j}\cdots(P_{j,r_j+1})_{B\Lambda_j}(e^{-B\Lambda_j(D_x)}\Psi_{j,\gamma}(D_x)v)\\ &= g_{j,R,\varepsilon,\gamma,B}(t,x)\end{aligned}$$

for $t \in [0,3\delta_1]$, where $(P_{j,k})_{B\Lambda_j} = (P_{j,k})_{B\Lambda_j}(t,x,D_t,D_x;R,\varepsilon)$ and

$$\begin{aligned}(2.35)\qquad g_{j,R,\varepsilon,\gamma,B}(t,x) =& e^{-B\Lambda_j}\Psi_{j,\gamma}g_{R,\varepsilon} - e^{-B\Lambda_j}R_j(t,x,D_t,D_x;R,\varepsilon)\Psi_{j,\gamma}v\\ &+ e^{-B\Lambda_j}[P_{R,\varepsilon},\Psi_{j,\gamma}]v.\end{aligned}$$

In §2.2 we shall derive microlocal energy estimates for the $(P_{j,k})_{B\Lambda_j}(t,x,D_t,D_x;R,\varepsilon)$.

2.2. Microlocal energy estimates

Define $\{v^k_{j,R,\varepsilon}\}_{1\le k\le r_j+1}$ for $1 \le j \le N_0$ by

$$(2.36)\qquad v^{r_j+1}_{j,R,\varepsilon} = e^{-B\Lambda_j(D_x)}\Psi_{j,\gamma}(D_x)v,$$

$$\begin{aligned}(2.37)\qquad v^{r_j+1-\mu}_{j,R,\varepsilon} &= (P_{j,r_j+2-\mu})_{B\Lambda_j}v^{r_j+2-\mu}_{j,R,\varepsilon}\\ &= (P_{j,r_j+2-\mu})_{B\Lambda_j}\cdots(P_{j,r_j+1})_{B\Lambda_j}v^{r_j+1}_{j,R,\varepsilon} \quad (\ 1 \le \mu \le r_j+1).\end{aligned}$$

Then (2.34) gives

$$(2.38)\qquad v^0_{j,R,\varepsilon} = g_{j,R,\varepsilon,\gamma,B}(t,x) \quad \text{for } t \in [0,3\delta_1].$$

We shall first derive microlocal energy estimates for $(P_{j,r_j+1})_{B\Lambda_j}(t,x,D_t,D_x;R,\varepsilon)$ ($1 \le j \le N_0$). If $m-2r_j = 0$ then $(P_{j,r_j+1})_{B\Lambda_j}(t,x,\tau,\xi;R,\varepsilon) = 1$. So we may assume that $m-2r_j > 0$. Fix $j \in \mathbf{N}$ so that $1 \le j \le N_0$. In this subsection we omit the subscript j of $P_{j,k}(\cdot)$, $b_{j,k}(\cdot)$, $\Lambda_j(\cdot)$, $\Psi_{j,\gamma}(\cdot)$, $\mathcal{C}_{j,\mu}$, r_j, $\cdots$ and so on, *i.e.*, we write $P_{j,k}(\cdot)$, $b_{j,k}(\cdot)$, $\Lambda_j(\cdot)$, $\Psi_{j,\gamma}(\cdot)$, $\mathcal{C}_{j,\mu}$, r_j, $\cdots$ as $P_k(\cdot)$, $b_k(\cdot)$, $\Lambda(\cdot)$, $\Psi_\gamma(\cdot)$, $\mathcal{C}_\mu$, r, $\cdots$, respectively. Then there is $\tilde{q}_{r+1}(t,x,\tau,\xi;R,\varepsilon,B) \in \mathcal{S}^{m-2r-1}_{1,0}$ uniformly in ε such that

$$(2.39)\qquad (P_{r+1})_{B\Lambda}(t,x,\tau,\xi;R,\varepsilon) = \tilde{p}_{r+1}(t,\tau,\xi) + \tilde{q}_{r+1}(t,x,\tau,\xi;R,\varepsilon,B).$$

Let $\psi(\xi) \in S^0_{1,0}$ satisfy

$$\psi(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{C}_3 \text{ and } |\xi| \ge 1/2, \\ 0 & \text{if } \xi \notin \mathcal{C}_4 \text{ or } |\xi| \le 1/4, \end{cases}$$

and put

$$\psi_\gamma(\xi) = (1 - \Theta_{\gamma/2}(\xi))\psi(\xi).$$

We define

$$\mathcal{E}_{r+1}(t; w, \gamma, l) = \sum_{\mu=1}^{m-2r} \|e^{-\gamma t}\langle D_x\rangle_\gamma^l p_{r+1,\mu}(t, D_t, D_x)\psi_\gamma(D_x)w\|^2_{L^2(\mathbf{R}^n_x)}$$

for $w(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n_x))$ with $w|_{t\le 0} = 0$, $t \in [0, 3\delta_1]$, $\gamma \ge 1$ and $l \in \mathbf{R}$, where the $p_{r+1,\mu}(t,\tau,\xi)$ are as in(2.17) with the subscript j omitted. Write

$$(f,g)_{L^2(\mathbf{R}^n)} \Big(\equiv (f,g)_{L^2(\mathbf{R}^n_x)} \Big) = \int_{\mathbf{R}^n} f(x)\overline{g(x)}\, dx.$$

A simple calculation yields

$$\begin{aligned} &D_t \mathcal{E}_{r+1}(t; w, \gamma, l) \\ &= \sum_{\mu=1}^{m-2r} \Big\{ 2i \operatorname{Im}((D_t - \lambda_{r+1,\mu}(t, D_x))p_{r+1,\mu}\psi_\gamma w, \\ &\qquad\qquad e^{-2\gamma t}\langle D_x\rangle_\gamma^{2l} p_{r+1,\mu}\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &\qquad + 2i\gamma \|e^{-\gamma t}\langle D_x\rangle^l_\gamma p_{r+1,\mu}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \Big\} \end{aligned} \tag{2.40}$$

for $t \in [0, 3\delta_1]$. Note that, for example,

$$\begin{aligned} &(\tau - \lambda_{r+1,\mu}(t,\xi)) \circ p_{r+1,\mu}(t,\tau,\xi)\psi_\gamma(\xi) \\ &= (\tau - \lambda_{r+1,\mu}(t,\xi)(1 - \Theta(4|\xi|))) \circ p_{r+1,\mu}(t,\tau,\xi)\psi_\gamma(\xi), \\ &\lambda_{r+1,\mu}(t,\xi)(1 - \Theta(4|\xi|)) \in S^1_{1,0}. \end{aligned}$$

We can write

$$\begin{aligned} &p_{r+1}(t,\tau,\xi) - i\partial_t p_{r+1,\mu}(t,\tau,\xi) \\ &= (P_{r+1})_{B\Lambda}(t,x,\tau,\xi;R,\varepsilon) - q_{r+1,\mu}(t,x,\tau,\xi;R,\varepsilon;B), \\ &q_{r+1,\mu}(t,x,\tau,\xi;R,\varepsilon;B) = (q_{r+1})_{B\Lambda}(t,x,\tau,\xi;R,\varepsilon) - i\partial_t p_{r+1,\mu}(t,\tau,\xi) \end{aligned}$$

for $(t,\xi) \in [0, 3\delta_1] \times \overline{\mathcal{C}}$ with $|\xi| \ge 1/4$ and $1 \le \mu \le m - 2r$, where $q_{r+1,\mu}(t,x,\tau,\xi;R,\varepsilon,B) \in \mathcal{S}^{m-2r-1}_{1,0}$ (uniformly in ε). Then we have

$$(\tau - \lambda_{r+1,\mu}(t,\xi)) \circ p_{r+1,\mu}(t,\tau,\xi)$$

$$= (P_{r+1})_{B\Lambda}(t,x,\tau,\xi;R,\varepsilon) - q_{r+1,\mu}(t,x,\tau,\xi;R,\varepsilon;B)$$

for $(t,\xi) \in [0,3\delta_1] \times \overline{\mathcal{C}}$ with $|\xi| \ge 1/4$ and $1 \le \mu \le m-2r$. This, together with (2.40), yields

$$\begin{aligned}(2.41)\quad & \partial_t \mathcal{E}_{r+1}(t;w,\gamma,l) \\ & \le \sum_{\mu=1}^{m-2r} \Big\{ \gamma^{-1} \| e^{-\gamma t} \langle D_x \rangle_\gamma^l (P_{r+1})_{B\Lambda}(t,x,D_t,D_x;R,\varepsilon)\psi_\gamma(D_x) w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \quad - \gamma \| e^{-\gamma t}\langle D_x\rangle_\gamma^l p_{r+1,\mu}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \quad + \gamma^{-1}\| e^{-\gamma t}\langle D_x\rangle_\gamma^l q_{r+1,\mu}(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \Big\}\end{aligned}$$

for $t \in [0,3\delta_1]$. Since $p_{r+1}(t,\tau,\xi)$ is strictly hyperbolic in τ for $(t,\xi) \in [0,3\delta_1] \times \overline{\mathcal{C}}$, it follows from Lagrange's interpolation theorem that there is $C(B) > 0$ such that

$$\begin{aligned}(2.42)\qquad & \sum_{\mu=1}^{m-2r} \| e^{-\gamma t}\langle D_x\rangle_\gamma^l q_{r+1,\mu}(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \le C(B) \sum_{\mu=1}^{m-2r} \| e^{-\gamma t}\langle D_x\rangle_\gamma^l p_{r+1,\mu}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\end{aligned}$$

for $\in [0,3\delta_1]$. By (2.41) and (2.42) there is $\gamma_{r+1}(B) \ge 1$ satisfying

$$\begin{aligned}& \partial_t \mathcal{E}_{r+1}(t;w,\gamma,l) \\ & \le (m-2r)\gamma^{-1} \| e^{-\gamma t}\langle D_x\rangle_\gamma^l (P_{r+1})_{B\Lambda}(t,x,D_t,D_x;R,\varepsilon)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\end{aligned}$$

for $l \in \mathbf{R}$, $t \in [0,3\delta_1]$ and $w(t,x) \in C^\infty(\mathbf{R};H^\infty(\mathbf{R}^n_x))$ with $w|_{t\le 0} = 0$, if $\gamma \ge \gamma_{r+1}(B)$. So we have

$$\mathcal{E}_{r+1}(t;w,\gamma,l) \le (m-2r)\gamma^{-1} \int_0^t \| e^{-\gamma s}\langle D_x\rangle_\gamma^l (P_{r+1})_{B\Lambda}\psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds$$

for $l \in \mathbf{R}$, $t \in [0,3\delta_1]$ and $w(t,x) \in C^\infty(\mathbf{R};H^\infty(\mathbf{R}^n_x))$ with $w|_{t\le 0} = 0$, if $\gamma \ge \gamma_{r+1}(B)$. By Lagrange's interpolation theorem $\tau^\nu \langle \xi\rangle_\gamma^l \psi_\gamma(\xi)$ ($\nu + l = m-2r-1$) can be represented by linear combinations of $\{p_{r+1,\mu}(t,\tau,\xi)\psi(\xi)\}_{1\le\mu\le m-2r}$ with symbols of (t,ξ) in $S^0_{1,0}(\mathbf{R}\times T^*\mathbf{R}^n)$ for $t \in [0,3\delta_1]$. Therefore, there is $C, C' > 0$ satisfying

$$(2.43)\qquad \sum_{\mu=0}^{m-2r-1} \| e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l+m-2r-1-\mu}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \le C' \mathcal{E}_{r+1}(t;w,\gamma,l)$$

$$\leq C\gamma^{-1}\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^l (P_{r+1})_{B\Lambda}\psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds$$

for $l \in \mathbf{R}$, $t \in [0, 3\delta_1]$ and $w(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n_x))$ with $w|_{t\leq 0} = 0$, if $\gamma \geq \gamma_{r+1}(B)$. Noting that $\psi_\gamma(\xi)\Psi_\gamma(\xi) = \Psi_\gamma(\xi)$, we have

$$(P_{r+1})_{B\Lambda}(\psi_\gamma v^{r+1}_{R,\varepsilon}) = \psi_\gamma v^r_{R,\varepsilon} + (1-\psi_\gamma)(P_{r+1})_{B\Lambda}(e^{-B\Lambda}\Psi_\gamma v).$$

Since $\operatorname{supp}\Psi_\gamma(\xi) \cap \operatorname{supp}(1-\psi_\gamma(\xi)) = \emptyset$, there is $R_{r+1}(t,x,\tau,\xi; R,\varepsilon,B,\gamma) \in \mathcal{S}^{m-2r-1,-\infty}_{1,0}$ uniformly in γ and ε satisfying

$$(P_{r+1})_{B\Lambda}(\psi_\gamma v^{r+1}_{R,\varepsilon}) = \psi_\gamma v^r_{R,\varepsilon} + R_{r+1}(t,x,D_t,D_x; R,\varepsilon,B,\gamma)\psi_\gamma v. \tag{2.44}$$

(2.43) with $w = v^{r+1}_{R,\varepsilon}$ and (2.44) yield

$$\begin{aligned}\sum_{\mu=0}^{m-2r-1} &\|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l+m-2r-1-\mu}\psi_\gamma v^{r+1}_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)} \\ &\leq C\gamma^{-1}\int_0^t \|e^{-\gamma s}\langle D_x\rangle^l_\gamma \psi_\gamma v^r_{R,\varepsilon}(s,x)\|^2_{L^2(\mathbf{R}^n_x)}\,ds \\ &\quad + C_{l,N}(B)\gamma^{-1}\sum_{\mu=0}^{m-2r-1}\int_0^t \|e^{-\gamma s}D_t^\mu\langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\end{aligned} \tag{2.45}$$

for $B \geq 1$, $l \in \mathbf{R}$, $t \in [0, 3\delta_1]$ and $\gamma \geq \gamma_{r+1}(B)$, where $C > 0$, $C_{l,N}(B) > 0$ and $N \in \mathbf{N}$. From (2.39) and (2.44) we have

$$\begin{aligned}D_t^{m-2r}\psi_\gamma v^{r+1}_{R,\varepsilon} = &-(\tilde{p}_{r+1}(t,D_t,D_x) - D_t^{m-2r})\psi_\gamma v^{r+1}_{R,\varepsilon} \\ &- \tilde{q}_{r+1}(t,x,D_t,D_x; R,\varepsilon,B)\psi_\gamma v^{r+1}_{R,\varepsilon} \\ &+ R_{r+1}(t,x,D_t,D_x; R,\varepsilon,B)\psi_\gamma v + \psi_\gamma v^r_{R,\varepsilon}.\end{aligned} \tag{2.46}$$

We can prove that there are $d^0_{r+1,\nu,l}(t,x,\tau,\xi; R,\varepsilon,B) \in \mathcal{S}^{m-2r-1,l-m+2r+1}_{1,0}$ uniformly in ε, $d^1_{r+1,\nu,l}(t,x,\tau,\xi; R,\varepsilon,B) \in \mathcal{S}^{\nu-m+2r,l-\nu}_{1,0}$ uniformly in ε and $R_{r+1,\nu,l}(t,x,\tau,\xi; R,\varepsilon,B,\gamma) \in \mathcal{S}^{\nu-1,-\infty}_{1,0}$ uniformly in γ and ε satisfying

$$\begin{aligned}D_t^\nu\langle D_x\rangle_\gamma^{l-\nu}\psi_\gamma v^{r+1}_{R,\varepsilon} = &d^0_{r+1,\nu,l}(t,x,D_t,D_x; R,\varepsilon,B)\psi_\gamma v^{r+1}_{R,\varepsilon} \\ &+ d^1_{r+1,\nu,l}(t,x,D_t,D_x; R,\varepsilon,B)\psi_\gamma v^r_{R,\varepsilon} \\ &+ R_{r+1,\nu,l}(t,x,D_t,D_x; R,\varepsilon,B,\gamma)\psi_\gamma v\end{aligned} \tag{2.47}$$

for $\nu \geq m-2r$ and $l \in \mathbf{R}$, by induction on ν. Indeed, from (2.46) we can see that (2.47) is valid for $\nu = m-2r$. Let $\kappa \in \mathbf{N}$ with $\nu \geq m-2r$, and suppose that (2.47) is valid for $\nu = \kappa$. Then we have

$$D_t^{\kappa+1}\langle D_x\rangle_\gamma^{l-\kappa-1}\psi_\gamma v^{r+1}_{R,\varepsilon} = d^0_{r+1,\kappa,l-1}D_t\psi_\gamma v^{r+1}_{R,\varepsilon} + [D_t, d^0_{r+1,\kappa,l-1}]\psi_\gamma v^{r+1}_{R,\varepsilon} \tag{2.48}$$

$$+ D_t d^1_{r+1,\kappa,l-1}\psi_\gamma v^r_{R,\varepsilon} + D_t R_{r+1,\kappa,l-1}\psi_\gamma v,$$

where $d^0_{r+1,\kappa,l-1} = d^0_{r+1,\kappa,l-1}(t,x,D_t,D_x;R,\varepsilon,B)$, $\cdots$ and so on. Note that $d^1_{r+1,m-2r,m-2r}(t,x,\tau,\xi;R,\varepsilon,B) = 1$. It follws from (2.47) with $\nu = l = m-2r$ and (2.48) that

$$\begin{aligned}
&D_t^{\kappa+1}\langle D_x\rangle_\gamma^{l-\kappa-1}\psi_\gamma v^{r+1}_{R,\varepsilon}\\
&\quad = d^0_{r+1,\kappa,l-1,m-2r-1}(d^0_{r+1,m-2r,m-2r}\psi_\gamma v^{r+1}_{R,\varepsilon} + \psi_\gamma v^r_{R,\varepsilon} + R_{r+1,m-2r,m-2r}\psi_\gamma v)\\
&\qquad + [D_t, d^0_{r+1,\kappa,l-1}]\psi_\gamma v^{r+1}_{R,\varepsilon} + \sum_{\mu=0}^{m-2r-2} d^0_{r+1,\kappa,l-1,\mu}D_t^{\mu+1}\psi_\gamma v^{r+1}_{R,\varepsilon}\\
&\qquad + D_t d^1_{r+1,\kappa,l-1}\psi_\gamma v^r_{R,\varepsilon} + D_t R_{r+1,\kappa,l-1}\psi_\gamma v,
\end{aligned}$$

where $d^0_{r+1,\kappa,l-1}(t,x,\tau,\xi;R,\varepsilon,B) = \sum_{\mu=0}^{m-2r-1} d^0_{r+1,\kappa,l-1,\mu}(t,x,\xi;R,\varepsilon,B)\tau^\mu$ and $d^0_{r+1,\kappa,l-1,\mu} = d^0_{r+1,\kappa,l-1,\mu}(t,x,D_x;R,\varepsilon,B)$. This implies that (2.47) is valid for $\nu = \kappa+1$. It follows from (2.36) and (2.47) that

$$\begin{aligned}
&\sum_{\mu=0}^{m} \|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu}e^{-B\Lambda}\Psi_\gamma v\|^2_{L^2(\mathbf{R}^n_x)}\\
&\le C_l(B)\Big\{\sum_{\mu=0}^{m-2r-1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu}e^{-B\Lambda}\Psi_\gamma v^{r+1}_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\\
&\qquad + \sum_{\mu=0}^{2r}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-m+2r-\mu}\psi_\gamma v^r_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\Big\}\\
&\qquad + C_{l,N}(B)\sum_{\mu=0}^{m-1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v\|^2_{L^2(\mathbf{R}^n_x)},
\end{aligned}$$

where $B \ge 1$, $l, N \in \mathbf{N}$ and $C_l(B)$ and $C_{l,N}(B)$ are positive constants. Therefore, this, together with (2.45) gives the following

Lemma 2.7. *There are positive constants $C_l(B)$ and $C_{l,N}(B)$ ($l \in \mathbf{R}$, $B \ge 1$, $N \in \mathbf{N}$) such that*

$$\begin{aligned}
&\sum_{\mu=0}^{m}\int_0^t \|e^{-\gamma s}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu}e^{-B\Lambda}\Psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\
&\le C_l(B)\sum_{\mu=0}^{2r}\int_0^t \|e^{-\gamma s}D_t^\mu\langle D_x\rangle_\gamma^{l-m+2r+1-\mu}\psi_\gamma v^r_{R,\varepsilon}|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\
&\qquad + C_{l,N}(B)\sum_{\mu=0}^{m-1}\int_0^t \|e^{-\gamma s}D_t^\mu\langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds
\end{aligned}$$

for $B \geq 1$, $l \in \mathbf{R}$, $N \in \mathbf{N}$, $t \in [0, 3\delta_1]$ and $\gamma \geq \gamma_{r+1}(B)$.

Remark. The lemma is well-known since $P_{r+1}(t, x, \tau, \xi; R, \varepsilon)$ is strictly hyperbolic in τ for $(t, x, \xi) \in [0, 3\delta_1] \times \mathbf{R}^n \times (\overline{\mathcal{C}} \setminus \{0\})$. To make the paper readable we gave the proof of the lemma.

Next we fix $k \in \mathbf{N}$ so that $1 \leq k \leq r$. Recall that

$$P_k(t, x, \tau, \xi; R, \varepsilon) = (\tau - b_k(t, \xi))^2 - a_k(t, \xi) + q_k(t, x, \tau, \xi; R, \varepsilon),$$
$$q_k(t, x, \tau, \xi; R, \varepsilon) = q^1_{k,0}(t, x, \xi; R, \varepsilon)\tau + q^1_{k,1}(t, x, \xi; R, \varepsilon) + q^0_k(t, x, \tau, \xi; R, \varepsilon)$$

for $(t, x, \tau, \xi) \in [-3\delta_1/2, 4\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times \overline{\mathcal{C}}$ with $|\xi| \geq 1/4$, where $q^1_{k,\mu}(t, x, \xi; R, \varepsilon) \in S^{\mu}_{1,0}(\mathbf{R} \times T^*\mathbf{R}^n)$ uniformly in ε ($\mu = 0, 1$) and $q^0_k(t, x, \tau, \xi; R, \varepsilon) \in \mathcal{S}^{1,-1}_{1,0}$ uniformly in ε. Therefore, there is $\tilde{q}_k(t, x, \tau, \xi; R, \varepsilon, B) \in \mathcal{S}^1_{1,0}$ uniformly in ε such that

$$(P_k)_{B\Lambda}(t, x, \tau, \xi; R, \varepsilon) = \tilde{p}_k(t, \tau, \xi) + \tilde{q}_k(t, x, \tau, \xi; R, \varepsilon, B). \tag{2.49}$$

Note that

$$\tilde{p}_k(t, \tau, \xi) = (\tau - b_k(t, \xi))^2 - a_k(t, \xi), \tag{2.50}$$

$$\begin{aligned} &\tilde{q}_k(t, x, \tau, \xi; R, \varepsilon, B) \\ &\quad = q^1_{k,0}(t, x, \xi; R, \varepsilon)\tau + q^1_{k,1}(t, x, \xi; R, \varepsilon) + \tilde{q}^0_k(t, x, \tau, \xi; R, \varepsilon, B) \end{aligned} \tag{2.51}$$

for $(t, x, \tau, \xi) \in [-3\delta_1/2, 4\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times \overline{\mathcal{C}}$ with $|\xi| \geq 1/4$, where $\tilde{q}^0_k(t, x, \tau, \xi; R, \varepsilon, B)/\log(1 + \langle\xi\rangle) \in \mathcal{S}^{1,-1}_{1,0}$ uniformly in ε. As $a_k(t, \xi)$ is real analytic in $[-3\delta_1/2, 4\delta_1] \times (\overline{\mathcal{C}} \setminus \{0\})$ and $a_k(t, \xi) \geq 0$, we can apply Lemma 2.2 (and its remark). Put

$$\tilde{\kappa}_k(\xi) = \int_0^{3\delta_1} a_k(t, \xi)\, dt \quad \text{for } \xi \in \overline{\mathcal{C}}.$$

Then there are $m_0 \in \mathbf{N}$ and $C > 0$ such that for any $\xi \in \overline{\mathcal{C}} \setminus \{0\}$ there are $m_k(\xi) \in \mathbf{Z}_+$ and $a_{k,\mu}(\xi) \in \mathbf{R}$ ($1 \leq \mu \leq m_k(\xi)$) satisfying $m_k(\xi) \leq m_0$ and

$$\begin{aligned} &C^{-1}\tilde{\kappa}_k(\xi)|t^{m_k(\xi)} + a_{k,1}(\xi)t^{m_k(\xi)-1} + \cdots + a_{k,m_k(\xi)}(\xi)| \\ &\quad \leq a_k(t, \xi) \leq C\tilde{\kappa}_k(\xi), \end{aligned} \tag{2.52}$$

$$|\partial_t a_k(t, \xi)| \leq C\tilde{\kappa}_k(\xi) \tag{2.53}$$

for $t \in [0, 3\delta_1]$, with a modification of δ_1 if necessary. Let $\widetilde{\Psi}(\xi)$ be a symbol in $S^0_{1,0}$ satisfying $0 \leq \widetilde{\Psi}(\xi) \leq 1$ and

$$\widetilde{\Psi}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{C}_4 \text{ and } |\xi| \geq 1/2, \\ 0 & \text{if } \xi \notin \mathcal{C} \text{ or } |\xi| \leq 1/4, \end{cases}$$

and define

$$[\![\, \xi \,]\!]_k = \sqrt{\tilde{\kappa}_k(\xi)\widetilde{\Psi}(\xi) + 1} \quad \text{for } \xi \in \mathbf{R}^n.$$

Lemma 2.8. *For $s \in \mathbf{R}$ and $\alpha \in (\mathbf{Z}_+)^n$ there is $C_{s,\alpha}$ satisfying*

$$|\partial^\alpha [\![\xi]\!]_k^s| \le C_{s,\alpha} [\![\xi]\!]_k^{s-|\alpha|}. \tag{2.54}$$

Proof. It is obvious that $[\![\xi]\!]_k^2 \in S^2_{1,0}$, $[\![\xi]\!]_k^2 \ge 1\,(\ge 0)$ and $|\partial^\alpha [\![\xi]\!]_k^2| \le C_{|\alpha|}\langle\xi\rangle^{2-|\alpha|}$. Fix $\mu \in \mathbf{N}$ with $1 \le \mu \le n$, and put $f(\xi) = [\![\xi]\!]_k^2$ and $e_\mu = (\delta_{\mu,1}, \cdots, \delta_{\mu,n}) \in \mathbf{R}^n$, where $\delta_{\mu,l} = 0$ if $\mu \ne l$ and $\delta_{\mu,\mu} = 1$. Then for any $h \in \mathbf{R}$ there is $\theta \in (0,1)$ satisfying

$$f(\xi + he_\mu) = f(\xi) + h\partial_{\xi_\mu} f(\xi) + \frac{h^2}{2}\partial^2_{\xi_\mu} f(\xi + \theta h e_\mu) \ge 1\,(\ge 0).$$

If $\pm h > 0$, we have

$$\mp\partial_{\xi_\mu} f(\xi) \le f(\xi)/|h| + |h|\partial^2_{\xi_\mu} f(\xi \pm \theta|h|e_\mu)/2.$$

Therefore, taking $|h| = \sqrt{2f(\xi)/C_2}\,(= \sqrt{2/C_2}\,[\![\xi]\!]_k)$ we have

$$|\partial_{\xi_\mu} [\![\xi]\!]_k^2| \le \sqrt{2C_2}\,[\![\xi]\!]_k.$$

If $|\alpha| = 1$, then we have

$$|\partial^\alpha [\![\xi]\!]_k^s| = |\partial^\alpha([\![\xi]\!]_k^2)^{s/2}| \le |s|\sqrt{C_2/2}\,[\![\xi]\!]_k^{s-1}.$$

Since $[\![\xi]\!]_k^2 \le C_0\langle\xi\rangle^2$ and $|\partial^\alpha [\![\xi]\!]_k^2| \le C_{|\alpha|}\langle\xi\rangle^{2-|\alpha|} \le C_{|\alpha|}C_0^{|\alpha|/2-1} [\![\xi]\!]_k^{2-|\alpha|}$ if $|\alpha| \ge 2$, there are $C'_\alpha > 0$ ($\alpha \in (\mathbf{Z}_+)^n$) satisfying

$$|\partial^\alpha [\![\xi]\!]_k^2| \le C'_\alpha [\![\xi]\!]_k^{2-|\alpha|} \quad (\alpha \in (\mathbf{Z}_+)^n).$$

Noting that

$$\partial^\alpha\partial_{\xi_\mu} [\![\xi]\!]_k^s = (s/2)\partial^\alpha\{([\![\xi]\!]_k^2)^{s/2-1}\partial_{\xi_\mu} [\![\xi]\!]_k^2\},$$

induction on $|\alpha|$ proves the lemma. □

We may assume that $m_0 \ge 2$. Define

$$\begin{aligned}
&\rho_0 = 2/(m_0+2),\\
&w_k(t,\xi) = a_k(t,\xi)\widetilde{\Psi}(\xi) + [\![\xi]\!]_k^{2\rho_0},\\
&W_{k,0}(t,\xi) = [\![\xi]\!]_k^{2\rho_0}\, w_k(t,\xi)^{-1/2} + 1,\\
&W_{k,1}(t,\xi)\\
&\quad = \Big(\sum_{\mu=1}^{r_0} \widetilde{\Psi}(\xi)^2|\beta^\mu(t,b_k(t,\xi),\xi)|^2|\xi|^{-2m+4} + [\![\xi]\!]_k^{2\rho_0}\Big)^{1/2} w_k(t,\xi)^{-1/2} + 1,
\end{aligned}$$

$$
\begin{aligned}
&W_{k,2,1}(t,\xi) = (\widetilde{\Psi}(\xi)^4|\partial_t a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{1/2}/w_k(t,\xi),\\
&W_{k,2,2}(t,\xi)\\
&\ = (\widetilde{\Psi}(\xi)^4|\partial_t\nabla_\xi a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{1/2}(\widetilde{\Psi}(\xi)^4|\nabla_\xi a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{-1/2},\\
&W_{k,2}(t,\xi) = W_{k,2,1}(t,\xi) + W_{k,2,2}(t,\xi) + 1
\end{aligned}
$$

for $(t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$, where the $\beta^\mu(t,\tau,\xi)$ are as in (2.22) and $\nabla_\xi f(\xi) = (\partial_{\xi_1} f(\xi), \cdots, \partial_{\xi_n} f(\xi))$. We also define the Riemannian metric $g_{k,\rho}$ on $\mathbf{R}^{2n}$ by

$$
g_{k,\rho\,(x,\xi)}(y,\eta) = |y|^2 + [\![\,\xi\,]\!]_k^{-2\rho}\,|\eta|^2,
$$

where $0 < \rho \le \rho_0$.

Lemma 2.9. *Let* $0 < \rho \le \rho_0$. (i) $g_{k,\rho}$ *is slowly varying and* $[\![\,\xi\,]\!]_k$ *is* $g_{k,\rho}$ *continuous, i.e., there are positive constants* c *and* C *such that*

$$
\begin{aligned}
&g_{k,\rho\,(x+y,\xi+\eta)}(X) \le C g_{k,\rho\,(x,\xi)}(X),\\
&C^{-1}\,[\![\,\xi\,]\!]_k \le [\![\,\xi+\eta\,]\!]_k \le C\,[\![\,\xi\,]\!]_k
\end{aligned}
$$

if (x,ξ), (y,η), $X \in \mathbf{R}^{2n}$ *and* $g_{k,\rho\,(x,\xi)}(y,\eta) \le c$. (ii)

$$
g^\sigma_{k,\rho\,(x,\xi)}(y,\eta)\left(\equiv \sup_X |\sigma((y,\eta),X)|^2/g_{k,\rho\,(x,\xi)}(X)\right) = [\![\,\xi\,]\!]_k^{2\rho}\,|y|^2 + |\eta|^2,
$$

where σ *denotes the symplectic form on* $\mathbf{R}^{2n}$. *Moreover,*

$$
h_{k,\rho}(x,\xi)\left(\equiv \left\{\sup_X g_{k,\rho\,(x,\xi)}(X)/g^\sigma_{k,\rho\,(x,\xi)}(X)\right\}^{1/2}\right) = [\![\,\xi\,]\!]_k^{-\rho} \le 1.
$$

(iii) $g_{k,\rho}$ *is* σ *temperate and* $[\![\,\xi\,]\!]_k$ *is* σ, $g_{k,\rho}$ *temperate.*

Proof. By Lemma 2.8 we have, with $C > 0$,

$$
|\,[\![\,\xi+\eta\,]\!]_k - [\![\,\xi\,]\!]_k\,| = \left|\eta \cdot \int_0^1 \nabla_\xi\,[\![\,\xi+\theta\eta\,]\!]_k\,d\theta\right| \le C|\eta|. \tag{2.55}
$$

Let $c > 0$, and assume that $g_{k,\rho\,(x,\xi)}(y,\eta) \le c$. Then we have $|\eta| \le \sqrt{c}\,[\![\,\xi\,]\!]_k^\rho$ $(\le \sqrt{c}\,[\![\,\xi\,]\!]_k)$. So, choosing $c \le c_0 \equiv (4C^2)^{-1}$, we have

$$
[\![\,\xi\,]\!]_k/2 \le [\![\,\xi+\eta\,]\!]_k \le 3\,[\![\,\xi\,]\!]_k/2. \tag{2.56}
$$

Since $2^{2\rho} \le 2$ and $(2/3)^{2\rho} \ge 2/3$, we have

$$
2g_{k,\rho\,(x,\xi)}(X)/3 \le g_{k,\rho\,(x+y,\xi+\eta)}(X) \le 2g_{k,\rho\,(x,\xi)}(X).
$$

This proves the assertion (i). The assertion (ii) is obvious. (2.55) gives

$$[\![\,\xi+\eta\,]\!]_k \le [\![\,\xi\,]\!]_k + C|\eta| \le C' [\![\,\xi\,]\!]_k (1+g^\sigma_{k,\rho\,(x,\xi)}(y,\eta))^{1/2},$$

which implies that

$$[\![\,\xi\,]\!]_k^{-1} \le C' [\![\,\xi+\eta\,]\!]_k^{-1} (1+g^\sigma_{k,\rho\,(x,\xi)}(y,\eta))^{1/2},$$
$$g_{k,\rho\,(x,\xi)}(X) \le C'' g_{k,\rho\,(x+y,\xi+\eta)}(X)(1+g^\sigma_{k,\rho\,(x,\xi)}(y,\eta))^{\rho},$$

where $C', C'' > 0$. This proves the assertion (iii). □

Lemma 2.10. *There are positive constants C, C_α and $C_{s,\alpha}$ ($s \in \mathbf{R}$, $\alpha \in (\mathbf{Z}_+)^n$) such that*

$$|\partial_t w_k(t,\xi)\widetilde{\Psi}(\xi)| \le W_{k,2}(t,\xi) w_k(t,\xi), \tag{2.57}$$
$$|\partial_\xi^\alpha w_k(t,\xi)^s| \le C_{s,\alpha} w_k(t,\xi)^s [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0} \quad (s \in \mathbf{R}), \tag{2.58}$$
$$|\partial_\xi^\alpha W_{k,0}(t,\xi)| \le C_\alpha W_{k,0}(t,\xi) [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}, \tag{2.59}$$
$$|\partial_\xi^\alpha W_{k,1}(t,\xi)| \le C_\alpha W_{k,1}(t,\xi) [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}, \tag{2.60}$$
$$|\partial_{\xi_\mu} W_{k,2}(t,\xi)| \le C W_{k,2}(t,\xi) [\![\,\xi\,]\!]_k^{-\rho_0} \quad (1 \le \mu \le n) \tag{2.61}$$

for $\alpha \in (\mathbf{Z}_+)^n$ and $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$.

Proof. (2.57) is obvious. Let $f(\xi) \in S^2_{1,0}$ satisfy $f(\xi) \ge 0$, and put

$$g(\xi) = \sqrt{f(\xi) + [\![\,\xi\,]\!]_k^{2\rho_0}}.$$

Then we have, with $C_\alpha > 0$ ($\alpha \in (\mathbf{Z}_+)^n$),

$$|\partial^\alpha g(\xi)| \le C_\alpha g(\xi) [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}. \tag{2.62}$$

Indeed, we can apply the same argument as in the proof of Lemma 2.8. In doing so, we use the fact that $\nabla_\xi f(\xi) = 0$ if $f(\xi) = 0$. Then we have, with $C > 0$,

$$|\partial_{\xi_\mu} f(\xi)| \le C\sqrt{f(\xi)} \quad \text{for } 1 \le \mu \le n. \tag{2.63}$$

Since $2\rho_0 - 1 \le 0$, we have

$$|\partial_{\xi_\mu} g(\xi)| = |\partial_{\xi_\mu} f(\xi) + \partial_{\xi_\mu} [\![\,\xi\,]\!]_k^{2\rho_0}|/(2g(\xi)) \le C' \le C' g(\xi) [\![\,\xi\,]\!]_k^{-\rho_0},$$

where $1 \le \mu \le n$ and $C' > 0$. This implies that (2.62) is valid for $|\alpha| = 1$. Let $l \in \mathbf{N}$, and suppose that (2.62) is valid for $|\alpha| \le l$. Let $|\alpha| = l+1$. Then, noting that

$$2g(\xi)\partial^\alpha g(\xi) + \sum_{0<\beta<\alpha} \binom{\alpha}{\beta} \partial^\beta g(\xi) \partial^{\alpha-\beta} g(\xi) = \partial^\alpha f(\xi) + \partial^\alpha [\![\,\xi\,]\!]_k^{2\rho_0},$$

we have, with $C'_\alpha > 0$,

$$(2.64)\quad 2g(\xi)|\partial^\alpha g(\xi)| \le C'_\alpha\Big(\langle\xi\rangle^{2-|\alpha|} + [\![\,\xi\,]\!]_k^{2\rho_0-|\alpha|} + \sum_{0<\beta<\alpha} g(\xi)^2\,[\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}\Big).$$

Since $\langle\xi\rangle^{2-|\alpha|} \le C''_\alpha\,[\![\,\xi\,]\!]_k^{(2-|\alpha|)\rho_0}$, $2\rho_0 - |\alpha| \le (2-|\alpha|)\rho_0$ and $[\![\,\xi\,]\!]_k^{(2-|\alpha|)\rho_0} \le g(\xi)^2\,[\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}$, (2.64) shows that (2.62) is valid for $|\alpha| = l+1$. (2.62), with induction on $|\alpha|$, gives

$$(2.65)\qquad |\partial^\alpha g(\xi)^s| \le C_{s,\alpha} g(\xi)^s\,[\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}\quad \text{for } \alpha\in(\mathbf{Z}_+)^n \text{ and } s\in\mathbf{R},$$

where the $C_{s,\alpha}$ are positive constants. (2.58) and (2.60) are simple consequences of (2.65) and (2.62), respectively. (2.59) easily follows from Lemma 2.8 and (2.58). Taking "$f(\xi) = \widetilde{\Psi}(\xi)^4|\partial_t\nabla_\xi a_k(t,\xi)|^2$ and $s=1$" and "$f(\xi) = \widetilde{\Psi}(\xi)^4|\nabla_\xi a_k(t,\xi)|^2$ and $s=-1$" in (2.65), respectively, we have

$$(2.66)\qquad |\partial^\alpha W_{k,2,2}(t,\xi)| \le C_\alpha W_{k,2,2}(t,\xi)\,[\![\,\xi\,]\!]_k^{-|\alpha|\rho_0}$$

for any $\alpha\in(\mathbf{Z}_+)^n$ and $(t,\xi)\in[0,3\delta_1]\times\mathbf{R}^n$, since $\widetilde{\Psi}(\xi)^4|\partial_t\nabla_\xi a_k(t,\xi)|^2$, $\widetilde{\Psi}(\xi)^4 \times |\nabla_\xi a_k(t,\xi)|^2 \in S^2_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$. A simple calculation yields

$$\begin{aligned}(2.67)\quad &|\partial_{\xi_\mu} W_{k,2,1}(t,\xi)|\\ &\le (|\widetilde{\Psi}(\xi)^2\partial_t a_k(t,\xi)\cdot\partial_{\xi_\mu}(\widetilde{\Psi}(\xi)^2\partial_t a_k(t,\xi))| + C\,[\![\,\xi\,]\!]_k^{2\rho_0-1})/w_k(t,\xi)\\ &\qquad\times(\widetilde{\Psi}(\xi)^4|\partial_t a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{-1/2} + W_{k,2,1}(t,\xi)\,[\![\,\xi\,]\!]_k^{-\rho_0}\\ &\le |\partial_{\xi_\mu}\widetilde{\Psi}(\xi)^2\cdot\partial_t a_k(t,\xi)|/w_k(t,\xi)\\ &\quad + (\widetilde{\Psi}(\xi)^4|\nabla_\xi a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{1/2} W_{k,2,2}(t,\xi)/w_k(t,\xi)\\ &\quad + C\,[\![\,\xi\,]\!]_k^{-1-\rho_0} + W_{k,2,1}(t,\xi)\,[\![\,\xi\,]\!]_k^{-\rho_0},\end{aligned}$$

where $1\le\mu\le n$, $(t,\xi)\in[0,3\delta_1]\times\mathbf{R}^n$ and $C>0$. (2.63) with $f(\xi) = \widetilde{\Psi}(\xi)a_k(t,\xi)$ gives

$$|\nabla_\xi(\widetilde{\Psi}(\xi)a_k(t,\xi))| \le C\sqrt{\widetilde{\Psi}(\xi)a_k(t,\xi)} \le C w_k(t,\xi)\,[\![\,\xi\,]\!]_k^{-\rho_0}.$$

Since $|\nabla_\xi\widetilde{\Psi}(\xi)| \le C\langle\xi\rangle^{-1}$, $\widetilde{\Psi}(\xi)a_k(t,\xi) \le C\langle\xi\rangle\sqrt{\widetilde{\Psi}(\xi)a_k(t,\xi)}$ and $w_k(t,\xi)^{1/2} \ge [\![\,\xi\,]\!]_k^{\rho_0}$, we have, with $C'>0$,

$$\begin{aligned}(2.68)\quad &|\widetilde{\Psi}(\xi)^2\nabla_\xi a_k(t,\xi)| \le |\nabla_\xi(\widetilde{\Psi}(\xi)a_k(t,\xi))| + |\widetilde{\Psi}(\xi)a_k(t,\xi)\nabla_\xi\widetilde{\Psi}(\xi)|\\ &\le C'\Big(w_k(t,\xi)\,[\![\,\xi\,]\!]_k^{-\rho_0} + \sqrt{\widetilde{\Psi}(\xi)a_k(t,\xi)}\Big) \le 2C' w_k(t,\xi)\,[\![\,\xi\,]\!]_k^{-\rho_0}.\end{aligned}$$

Next we shall consider $\widetilde{\Psi}(\xi)\partial_t a_k(t,\xi)$. Put $q(t) = \widetilde{\Psi}(\xi)a_k(t,\xi)$ for $(t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$. Recall that $\widetilde{\Psi}(\xi)a_k(t,\xi)$ is defined for $(t,\xi) \in [-2\delta_1, 4\delta_1] \times \mathbf{R}^n$. Then, for $h \in [-\delta_1, \delta_1]$ there is $\theta \in (0,1)$ such that

$$0 \le q(t+h) = q(t) + hq'(t) + h^2 q''(t+\theta h)/2$$

for $t \in [0, 3\delta_1]$. Therefore, we have

$$\pm q'(t) \le q(t)/|h| + |h|q''(t+\theta h)/2.$$

This gives, with $C_0 > 0$,

$$|\widetilde{\Psi}(\xi)\partial_t a_k(t,\xi)| \le \widetilde{\Psi}(\xi)a_k(t,\xi)/|h| + C_0|h|\widetilde{\Psi}(\xi)\langle\xi\rangle^2 \tag{2.69}$$

for $(t,\xi) \in [0,3\delta_1]$ and $h \in [-\delta_1,\delta_1]$. When $\sqrt{a_k(t,\xi)\langle\xi\rangle^{-2}/C_0} \le \delta_1$, taking $|h| = \sqrt{a_k(t,\xi)\langle\xi\rangle^{-2}/C_0}$ we have

$$|\widetilde{\Psi}(\xi)\partial_t a_k(t,\xi)| \le 2\sqrt{C_0 a_k(t,\xi)}\widetilde{\Psi}(\xi)\langle\xi\rangle \tag{2.70}$$

for $(t,\xi) \in [0,3\delta_1]$. Note that (2.70) is still valid when $a_k(t,\xi) = 0$. When $\sqrt{a_k(t,\xi)\langle\xi\rangle^{-2}/C_0} \ge \delta_1$, taking $|h| = \delta_1$ in (2.69), we have

$$\begin{aligned}
&C_0\delta_1^2\widetilde{\Psi}(\xi)\langle\xi\rangle^2 \le \widetilde{\Psi}(\xi)a_k(t,\xi) \le C\widetilde{\Psi}(\xi)\langle\xi\rangle^2, \\
&|\widetilde{\Psi}(\xi)\partial_t a_k(t,\xi)| \le \widetilde{\Psi}(\xi)a_k(t,\xi)/\delta_1 + C_0\delta_1\widetilde{\Psi}(\xi)\langle\xi\rangle^2 \\
&\le C'\sqrt{a_k(t,\xi)}\widetilde{\Psi}(\xi)\langle\xi\rangle/\delta_1.
\end{aligned}$$

This, together with (2.70), gives

$$|\partial_{\xi_\mu}\widetilde{\Psi}(\xi)^2 \cdot \partial_t a_k(t,\xi)|/w_k(t,\xi) \le C'' w_k(t,\xi)^{-1/2}/\delta_1 \le C'' [\![\xi]\!]_k^{-\rho_0}/\delta_1 \tag{2.71}$$

for $1 \le \mu \le n$ and $(t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$, where $C'' > 0$. Therefore, (2.61) follows from (2.66) – (2.68) and (2.71). □

Lemma 2.11. *Let $\rho > 0$, and let $f(\xi) \in C^1(\mathbf{R}^n)$ satisfy $f(\xi) > 0$ and*

$$|\partial_{\xi_\mu} f(\xi)| \le C(f)f(\xi)[\![\xi]\!]_k^{-\rho} \quad \text{for } 1 \le \mu \le n \text{ and } \xi \in \mathbf{R}^n. \tag{2.72}$$

Then, for any $\delta > 0$ there is $c_\delta \equiv c_\delta(C(f)) > 0$ satisfying

$$(1+\delta)^{-1} \le f(\eta)/f(\xi) \le 1+\delta \tag{2.73}$$

if $\xi, \eta \in \mathbf{R}^n$ and $|\xi - \eta| \le \sqrt{c_\delta}[\![\xi]\!]_k^\rho$. In particular, $f(\xi)$ is $g_{k,\rho}$ continuous.

Proof. Let $0 < c_\delta \le c_0$ and $|\xi - \eta| \le \sqrt{c_\delta}\, [\![\xi]\!]_k^\rho$, where c_0 is the constant in (2.56). Then, by (2.56) and (2.72) we have

$$\begin{aligned}
&|\log f(\xi) - \log f(\eta)| \\
&= \Big| \sum_{\mu=1}^{n} \int_0^1 \partial_{\xi_\mu} f(\xi + \theta(\eta - \xi)) \cdot (\eta_\mu - \xi_\mu) f(\xi + \theta(\eta - \xi))^{-1}\, d\theta \Big| \\
&\le 2^\rho C(f) \sum_{\mu=1}^{n} \int_0^1 |\xi_\mu - \eta_\mu|\, [\![\xi]\!]_k^{-\rho}\, d\theta \le 2^\rho C(f) \sqrt{n} |\xi - \eta|\, [\![\xi]\!]_k^{-\rho} \\
&\le 2^\rho C(f) \sqrt{n c_\delta}, \\
&\exp[-2^\rho C(f) \sqrt{n c_\delta}] \le f(\eta)/f(\xi) \le \exp[2^\rho C(f) \sqrt{n c_\delta}].
\end{aligned}$$

Taking $c_\delta = \min\{c_0, (\log(1+\delta))^2/(2^{2\rho} n C(f)^2)\}$, we obtain (2.73). □

Lemma 2.12. (i) *For any $\delta > 0$ there is $c'_\delta > 0$ such that*

$$\begin{aligned}
&(1+\delta)^{-1} \le w_k(t,\eta)/w_k(t,\xi) \le 1 + \delta, \\
&(1+\delta)^{-1} \le W_{k,\mu}(t,\eta)/W_{k,\mu}(t,\xi) \le 1 + \delta \quad (\, 0 \le \mu \le 2)
\end{aligned}$$

if $\xi, \eta \in \mathbf{R}^n$, $t \in [0, 3\delta_1]$ and $|\xi - \eta| \le \sqrt{c'_\delta}\, [\![\xi]\!]_k^{\rho_0}$. Moreover, there is $C > 0$ such that

$$w_k(t,\xi) \le C\, [\![\xi]\!]_k^2, \tag{2.74}$$
$$W_{k,0}(t,\xi) \le 2\, [\![\xi]\!]_k^{\rho_0}, \tag{2.75}$$
$$W_{k,1}(t,\xi) \le C\, [\![\xi]\!]_k^{1-\rho_0}, \tag{2.76}$$
$$W_{k,2}(t,\xi) \le C\, [\![\xi]\!]_k^{2-\rho_0}, \tag{2.77}$$
$$\widetilde{\Psi}(\xi)^2 |\nabla_\xi a_k(t,\xi)| \le C\, [\![\xi]\!]_k \tag{2.78}$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$. (ii) *$W_{k,1}(t,\xi)$ is uniformly σ, g_{k,ρ_0} temperate in $t \in [0, 3\delta_1]$.* (iii) *Modifying δ_1 and m_0 if necessary, for $l = 0, 1$ we have*

$$\begin{aligned}
&\#\{t \in [0, 3\delta_1];\ \partial_t \partial_{\xi_\mu}^l a_k(t,\xi) = 0\} \le m_0 \\
&\qquad \text{if } 1 \le \mu \le n,\ \xi \in \overline{\mathcal{C}} \cap S^{n-1} \text{ and } \partial_t \partial_{\xi_\mu}^l a_k(t,\xi) \not\equiv 0 \text{in } t.
\end{aligned}$$

Proof. The first part of the assertion (i) easily follows from Lemmas 2.10 and 2.11. (2.52) proves (2.74). (2.75) is obvious. Let us prove (2.76). From the definitions of the $\beta^\mu(t,\tau,\xi)$ we have

$$\min\{ \min_{s \in \mathcal{R}(\xi/|\xi|)} |t - s|^2, 1\} |\beta^\mu(t, b_k(t,\xi), \xi)|^2 \le C h_{m-1}(t, b_k(t,\xi), \xi) \tag{2.79}$$

for $(t,\xi) \in [0,3\delta_1] \times (\overline{\mathcal{C}} \setminus \{0\})$. By (1.1) there is $C>0$ satisfying

$$C^{-1} a_k(t,\xi)|\xi|^{2m-4} \le h_{m-1}(t,b_k(t,\xi),\xi) \le C a_k(t,\xi)|\xi|^{2m-4} \tag{2.80}$$

for $(t,\xi) \in [0,3\delta_1] \times (\overline{\mathcal{C}} \setminus \{0\})$. Therefore, we have

$$\min\{\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|^2, 1\}|\beta^\mu(t,b_k(t,\xi),\xi)|^2/|\xi|^{2m-4} \le C'\tilde{\kappa}_k(\xi). \tag{2.81}$$

Now we use the notations in the proof of Lemma 2.2 with $\kappa(\xi)$ replaced by $\tilde{\kappa}_k(\xi)$ (see, also, the remark of Lemma 2.2). Let $\xi^0 \in \overline{\mathcal{C}} \cap S^{n-1}$ and $p \in \widetilde{U}(\xi^0)$. Then we have

$$\tilde{\kappa}_k(\varphi(\tilde{u})) = e(X(\tilde{u})) \prod_{\mu=1}^{r(p)} X_\mu(\tilde{u})^{2s_\mu(p)} \quad (\tilde{u} \in \widetilde{U}(\xi^0;p)),$$

where $e(X)>0$ for $X \in \widetilde{V}(\xi^0;p)$. So we have, with $C''>0$,

$$\min\{\min_{s\in\mathcal{R}(\tilde{\varphi}(X)/|\tilde{\varphi}(X)|)} |t-s|^2, 1\}|\beta^\mu(t,b_k(t,\tilde{\varphi}(X)),\tilde{\varphi}(X))|^2 \le C''\tilde{\kappa}_k(\tilde{\varphi}(X))$$

for $t \in [0,3\delta_1]$ and $X \in \widetilde{V}(\xi^0;p)$. This implies that

$$\beta^\mu(t,b_k(t,\tilde{\varphi}(X)),\tilde{\varphi}(X))^2 = \tilde{\beta}^\mu(t,X)\tilde{\kappa}_k(\tilde{\varphi}(X)),$$

where $\tilde{\beta}^\mu(t,X)$ is real analytic in (t,X). Therefore, we have

$$|\beta^\mu(t,b_k(t,\xi),\xi)| \le C\,[\![\,\xi\,]\!]_k\,|\xi|^{m-2} \tag{2.82}$$

for $(t,\xi) \in [0,3\delta_1] \times \overline{\mathcal{C}}$ with $|\xi| \ge 1$, which proves (2.76). (2.63) gives

$$\begin{aligned} &|\widetilde{\Psi}(\xi)^2\nabla_\xi a_k(t,\xi)| \le |\nabla_\xi(\widetilde{\Psi}(\xi)a_k(t,\xi))| + |\widetilde{\Psi}(\xi)a_k(t,\xi)\nabla_\xi\widetilde{\Psi}(\xi)| \\ &\le C\sqrt{\widetilde{\Psi}(\xi)a_k(t,\xi)}, \end{aligned}$$

which proves (2.78). It follows from Lemma 2.2 and (2.78) that

$$\widetilde{\Psi}(\xi)^2|\partial_t\nabla_\xi a_k(t,\xi)| \le C\Big(\int_0^{3\delta_1} \widetilde{\Psi}(\xi)^4|\nabla_\xi a_k(t,\xi)|^2\,dt\Big)^{1/2} \le C'\delta_1\,[\![\,\xi\,]\!]_k$$

for $(t,\xi) \in [0,3\delta_1] \times \overline{\mathcal{C}}$. This, together with (2.53), proves (2.77). The assertion (i) implies that $W_{k,1}(t,\xi)$ is uniformly g_{k,ρ_0} continuous in t. If $\xi,\eta \in \mathbf{R}^n$ and $|\xi-\eta| \le \sqrt{c_1'}\,[\![\,\xi\,]\!]_k^{\rho_0}$, then we have

$$W_{k,1}(t,\xi) \le W_{k,1}(t,\eta)(1+g^\sigma_{k,\rho\,(x,\xi)}(y-x,\eta-\xi))$$

for $t \in [0, 3\delta_1]$, $x, y \in \mathbf{R}^n$ and $\rho > 0$, where c_1' is the constant in Lemma 2.12 with $\delta = 1$. Suppose that $\xi, \eta \in \mathbf{R}^n$, $|\xi - \eta| \geq \sqrt{c_1'}\,[\![\, \xi \,]\!]_k^{\rho_0}$ and $\rho > 0$. We may assume that $c_1' \leq 1$. Then we have

$$1 + g_{k,\rho\,(x,\xi)}(y - x, \eta - \xi) \geq c_1'(1 + [\![\, \xi \,]\!]_k^{2\rho_0}).$$

This, together with (2.76), gives

$$W_{k,1}(t,\xi) \leq C\,[\![\, \xi \,]\!]_k^{1-\rho_0} \leq C' W_{k,1}(t,\eta)(1 + g^{\sigma}_{k,\rho\,(x,\xi)}(y - x, \eta - \xi))^{(1-\rho_0)/(2\rho_0)},$$

which proves the assertion (ii). The assertion (iii) easily follows from Lemma 2.2, since $\partial_t a_k(t,\xi)$ and $\partial_t \partial_{\xi_\mu} a_k(t,\xi)$ are real analytic. □

Let $\theta(\xi) \in C_0^\infty(\mathbf{R}^n)$ satisfy $\theta(\xi) \geq 0$, $\int_{\mathbf{R}^n} \theta(\xi)\, d\xi = 1$ and $\operatorname{supp} \theta \subset \{\xi \in \mathbf{R}^n;\ |\xi| \leq \sqrt{c_1'}\}$. We define

$$\widetilde{W}_{k,2}(t,\xi) = \int_{\mathbf{R}^n} \theta([\![\, \eta \,]\!]_k^{-\rho_0}(\xi - \eta)) W_{k,2}(t,\eta)\,[\![\, \eta \,]\!]_k^{-n\rho_0}\, d\eta$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$. Then we can prove the following lemma, applying the same argument as in Lemma 3.4 of [12].

Lemma 2.13. *Modifying c_1' if necessary, we have, with $C_\alpha > 0$ ($\alpha \in (\mathbf{Z}_+)^n$),*

$$W_{k,2}(t,\xi)/4 \leq \widetilde{W}_{k,2}(t,\xi) \leq 4W_{k,2}(t,\xi),$$
$$|\partial_\xi^\alpha \widetilde{W}_{k,2}(t,\xi)| \leq C_\alpha \widetilde{W}_{k,2}(t,\xi)\,[\![\, \xi \,]\!]_k^{-|\alpha|\rho_0}$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$ and $\alpha \in (\mathbf{Z}_+)^n$.

Define

$$\Phi_k(t,\xi) = \int_0^t (W_{k,0}(s,\xi) + W_{k,1}(s,\xi) + \widetilde{W}_{k,2}(s,\xi))\, ds$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$.

Lemma 2.14. *There are $C_\alpha > 0$ ($\alpha \in (\mathbf{Z}_+)^n$) such that*

$$|\partial_\xi^\alpha \Phi_k(t,\xi)| \leq C_\alpha (1 + \log\,[\![\, \xi \,]\!]_k)\,[\![\, \xi \,]\!]_k^{-|\alpha|\rho_0}$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$ and $\alpha \in (\mathbf{Z}_+)^n$.

Proof. By Lemmas 2.10 and 2.13 it suffices to prove that

$$(0 \le)\Phi_k(3\delta_1,\xi) \le C_0(1+\log [\![\,\xi\,]\!]_k) \quad \text{for } \xi \in \mathbf{R}^n. \tag{2.83}$$

If $\tilde{\kappa}_k(\xi)\widetilde{\Psi}(\xi) = 0$, then we have $[\![\,\xi\,]\!]_k = 1$, $w_k(t,\xi) = 1$, $W_{k,0}(t,\xi) = 2$, $W_{k,1}(t,\xi) = 2$ and $W_{k,2}(t,\xi) = 2$ by (2.63) and (2.81). So (2.83) holds if $\tilde{\kappa}_k(\xi)\widetilde{\Psi}(\xi) = 0$. Now assume that $\xi \in \mathbf{R}^n$ and $\tilde{\kappa}_k(\xi)\widetilde{\Psi}(\xi) > 0$. It follows from (2.52) that there is $c_0 > 0$ satisfying

$$w_k(t,\xi) \ge c_0\Big([\![\,\xi\,]\!]_k^2 \prod_{\mu=1}^{m_k(\xi)} |t-t_\mu(\xi)| + [\![\,\xi\,]\!]_k^{2\rho_0}\Big),$$

where $t^{m_k(\xi)} + a_{k,1}(\xi)t^{m_k(\xi)-1} + \cdots + a_{k,m_k(\xi)}(\xi) = \prod_{\mu=1}^{m_k(\xi)}(t-t_\mu(\xi))$. So there is $C > 0$ such that

$$\int_0^{3\delta_1} w_k(t,\xi)^{-1/2}\,dt \le C\int_0^{3\delta_1} [\![\,\xi\,]\!]_k^{-1}\Big(\prod_{\mu=1}^{m_k(\xi)} |t-t_\mu(\xi)| + [\![\,\xi\,]\!]_k^{2\rho_0-2}\Big)^{-1/2}\,dt$$

Write

$$\begin{aligned}&(\{0,3\delta_1\} \cup \{\operatorname{Re} t_\mu(\xi)\}_{1\le\mu\le m_k(\xi)}) \cap [0,3\delta_1] = \{t_0,t_1,\cdots,t_{m_k'(\xi)+1}\},\\ &t_{-1} = 0 = t_0 < t_1 < t_2 < \cdots < t_{m_k'(\xi)+1} = 3\delta_1 = t_{m_k'(\xi)+2}.\end{aligned}$$

Then we have, with $C', C'' > 0$,

$$\begin{aligned}&\int_0^{3\delta_1} w_k(t,\xi)^{-1/2}\,dt\\ &\le C\sum_{\mu=0}^{m_k'(\xi)+1}\int_{(t_\mu+t_{\mu-1})/2}^{(t_\mu+t_{\mu+1})/2} [\![\,\xi\,]\!]_k^{-1}\Big(\prod_{\nu=1}^{m_k(\xi)} |t-t_\nu(\xi)| + [\![\,\xi\,]\!]_k^{2\rho_0-2}\Big)^{-1/2}\,dt\\ &\le C'\sum_{\mu=0}^{m_k'(\xi)+1}\int_0^{3\delta_1} [\![\,\xi\,]\!]_k^{-1}(|t-t_\mu| + [\![\,\xi\,]\!]_k^{2(\rho_0-1)/m_0})^{-m_0/2}\,dt\\ &\le \begin{cases} C''\,[\![\,\xi\,]\!]_k^{-2\rho_0} & \text{if } m_0 > 2,\\ C''\,[\![\,\xi\,]\!]_k^{-2\rho_0}(1+\log [\![\,\xi\,]\!]_k) & \text{if } m_0 = 2,\end{cases}\end{aligned}$$

since $m_k(\xi) \le m_0$ and $2(\rho_0-1)/m_0 \cdot (2-m_0)/2 - 1 = -2\rho_0$. This gives, with $C > 0$,

$$\int_0^{3\delta_1} W_{k,0}(t,\xi)\,dt \le C(1+\log [\![\,\xi\,]\!]_k). \tag{2.84}$$

We note that (2.84) was proved in [2] when $a_k(t,\xi)$ is a non-negative quadratic form of ξ. By (2.76), (2.79) and (2.80), we have, with $C>0$,

$$\int_0^{3\delta_1} W_{k,1}(t,\xi)\,dt \le C\int_0^{3\delta_1} \min\Big\{\Big(\min_{s\in\mathcal{R}(\xi/|\xi|)}|t-s|\Big)^{-1}, [\![\,\xi\,]\!]_k^{1-\rho_0}\Big\}\,dt.$$

Write

$$\begin{aligned}&(\{0,3\delta_1\}\cup\{\mathrm{Re}\,\lambda;\ \lambda\in\mathcal{R}(\xi/|\xi|)\})\cap[0,3\delta_1]=\{\tau_0,\tau_1,\cdots,\tau_{N(\xi)}\},\\ &0=\tau_0<\tau_1<\tau_2<\cdots<\tau_{N(\xi)}=3\delta_1=\tau_{N(\xi)+1}.\end{aligned}$$

Note that $N(\xi)\le N_{3\delta_1}+1$, where $N_{3\delta_1}$ is the integer in (1.2). Put $\tilde\tau_0=0$ and $\tilde\tau_{\mu+1}=(\tau_\mu+\tau_{\mu+1})/2$, $\tau_\mu^-=\max\{\tau_\mu-[\![\,\xi\,]\!]_k^{\rho_0-1},\tilde\tau_\mu\}$ and $\tau_\mu^+=\min\{\tau_\mu+[\![\,\xi\,]\!]_k^{\rho_0-1},\tilde\tau_{\mu+1}\}$ ($0\le\mu\le N(\xi)$), we have, with $C,C'>0$,

$$\begin{aligned}\int_0^{3\delta_1} W_{k,1}(t,\xi)\,dt \le C\sum_{\mu=0}^{N(\xi)}\Big\{\int_{\tilde\tau_\mu}^{\tau_\mu^-}(\tau_\mu-t)^{-1}\,dt+\int_{\tau_\mu^-}^{\tau_\mu^+}[\![\,\xi\,]\!]_k^{1-\rho_0}\,dt\\ +\int_{\tau_\mu^+}^{\tilde\tau_{\mu+1}}(t-\tau_\mu)^{-1}\,dt\Big\}\le C'(1+\log[\![\,\xi\,]\!]_k).\end{aligned}$$

Let $N_0\in\mathbf{Z}_+$ and $p\in\mathbf{R}$, and let $f(t,\xi)$ be a function defined for $(t,\xi)\in[0,3\delta_1]\times\mathcal{C}$ satisfying the following:

(i) $f(t,\xi)$ is continuously differentiable in $t\in[0,3\delta_1]$.

(ii) $\#\{t\in[0,3\delta_1];\,\partial_t f(t,\xi)=0\}\le N_0$ if $\xi\in\mathcal{C}$ and $\partial_t f(t,\xi)\not\equiv 0$ in t.

(iii) $|f(t,\xi)|\le C_0\,[\![\,\xi\,]\!]_k^p$ for $(t,\xi)\in[0,3\delta_1]\times\mathcal{C}$ with $|\xi|\ge 1$.

Then there is $C(N_0,C_0,p)>0$ such that

$$\int_0^{3\delta_1}|\partial_t f(t,\xi)|/(|f(t,\xi)|+1)\,dt\le C(N_0,C_0,p)(1+\log[\![\,\xi\,]\!]_k) \tag{2.85}$$

for $\xi\in\mathcal{C}$ with $|\xi|\ge 1$. Indeed, (2.85) is obvious if $\partial_t f(t,\xi)\equiv 0$ in t, where $\xi\in\mathcal{C}$. Fix $\xi\in\mathcal{C}$ so that $|\xi|\ge1$ and $\partial_t f(t,\xi)\not\equiv 0$ in t. Write

$$\begin{aligned}&\{t\in[0,3\delta_1];\ f(t,\xi)\partial_t f(t,\xi)=0\}=\{t_1,t_2,\cdots,t_{N(\xi)}\},\\ &0\le t_1<t_2<\cdots<t_{N(\xi)}\le 3\delta_1.\end{aligned}$$

It is obvious that $N(\xi) \le 2N_0 + 1$. In each subinterval $[t_{\mu-1}, t_\mu]$ ($1 \le \mu \le N(\xi) + 1$) we have "$f(t,\xi) \ge 0$ or $f(t,\xi) \le 0$" and "$\partial_t f(t,\xi) \ge 0$ or $\partial_t f(t,\xi) \le 0$", where $t_0 = 0$ and $t_{N(\xi)+1} = 3\delta_1$. Then we have

$$\int_{t_{\mu-1}}^{t_\mu} |\partial_t f(t,\xi)|/(|f(t,\xi)|+1)\,dt = |\log\{(|f(t_\mu,\xi)|+1)/(|f(t_{\mu-1},\xi)|+1)\}|$$
$$\le 2\log(C_0 [\![\,\xi\,]\!]_k^p + 1) \le \begin{cases} 2\log(C_0+1) + 2p\log [\![\,\xi\,]\!]_k & \text{if } p \ge 0, \\ 2\log(C_0+1) & \text{if } p \le 0, \end{cases}$$

which proves (2.85). Note that

$$\begin{aligned} W_{k,2}(t,\xi) &\le (\widetilde{\Psi}(\xi)^2|\partial_t a_k(t,\xi)| + [\![\,\xi\,]\!]_k^{\rho_0})/(\widetilde{\Psi}(\xi)a_k(t,\xi) + [\![\,\xi\,]\!]_k^{2\rho_0}) \\ &\quad + 2\sum_{\mu=1}^n (\widetilde{\Psi}(\xi)^2|\partial_t\partial_{\xi_\mu} a_k(t,\xi)| + [\![\,\xi\,]\!]_k^{\rho_0})/(\widetilde{\Psi}(\xi)^2|\partial_{\xi_\mu} a_k(t,\xi)| + [\![\,\xi\,]\!]_k^{\rho_0}) \\ &\le 2|\partial_t(\widetilde{\Psi}(\xi)a_k(t,\xi) + [\![\,\xi\,]\!]_k^{2\rho_0})|/(\widetilde{\Psi}(\xi)a_k(t,\xi) + [\![\,\xi\,]\!]_k^{2\rho_0} + 1) + [\![\,\xi\,]\!]_k^{-\rho_0} \\ &\quad + 4\sum_{\mu=1}^n |\partial_t(\widetilde{\Psi}(\xi)^2\partial_{\xi_\mu} a_k(t,\xi) + [\![\,\xi\,]\!]_k^{\rho_0})|/(\widetilde{\Psi}(\xi)^2|\partial_{\xi_\mu} a_k(t,\xi)| + [\![\,\xi\,]\!]_k^{\rho_0} + 1) + 2n. \end{aligned}$$

This, together with (2.85), gives, with $C > 0$,

$$\int_0^{3\delta_1} W_{k,2}(t,\xi)\,dt \le C(1 + \log [\![\,\xi\,]\!]_k),$$

which proves the lemma. □

Let $A > 0$, $\gamma \ge 1$ and $l \in \mathbf{R}$. We define

$$\begin{aligned} &K_k(t,\xi;A,\gamma,l) = e^{-2\gamma t}\widetilde{K}_k(t,\xi;A,\gamma,l), \\ &\widetilde{K}_k(t,\xi;A,\gamma,l) = \exp[-A\Phi_k(t,\xi) - 2t\log\langle\xi\rangle_\gamma + 2l\log\langle\xi\rangle_\gamma]. \end{aligned}$$

Fix ρ so that $0 < \rho < \rho_0$.

Lemma 2.15. *There are $C_\alpha(A,l) > 0$ ($\alpha \in (\mathbf{Z}_+)^n$) such that*

$$|\partial_\xi^\alpha \widetilde{K}_k(t,\xi;A,\gamma,l)| \le C_\alpha(A,l)\widetilde{K}_k(t,\xi;A,\gamma,l)\,[\![\,\xi\,]\!]_k^{-|\alpha|\rho} \tag{2.86}$$

for $\alpha \in (\mathbf{Z}_+)^n$ and $(t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$. Moreover, $\widetilde{K}_k(t,\xi;A,\gamma,l)$ is uniformly σ, $g_{k,\rho}$ temperate in ε and $t \in [0,3\delta_1]$.

Proof. Since

$$\partial_{\xi_\mu}\widetilde{K}_k(t,\xi;A,\gamma,l) = (-A\partial_{\xi_\mu}\Phi_k(t,\xi) + (2l-2t)\xi_\mu\langle\xi\rangle_\gamma^{-2})\widetilde{K}_k(t,\xi;A,\gamma,l),$$

Lemma 2.14 and induction on $|\alpha|$ prove (2.86). From Lemma 2.11 $\widetilde{K}_k(t,\xi;A,\gamma,l)$ is uniformly $g_{k,\rho}$ continuous, *i.e.*, there is $c(A,l)>0$ satisfying

$$1/2 \le \widetilde{K}_k(t,\xi;A,\gamma,l)/\widetilde{K}_k(t,\eta;A,\gamma,l) \le 2$$

if $t\in[0,3\delta_1]$, $\xi,\eta\in\mathbf{R}^n$ and $|\xi-\eta|\le c(A,l)\,[\![\,\xi\,]\!]_k^\rho$. Suppose that $\xi,\eta\in\mathbf{R}^n$ and $|\xi-\eta|\ge c(A,l)\,[\![\,\xi\,]\!]_k^\rho$. Noting that

$$\begin{aligned}&\langle\xi\rangle_\gamma^{2l-2t}\exp[-AC_0(1+\log[\![\,\xi\,]\!]_k)] \le \widetilde{K}_k(t,\xi;A,\gamma,l) \le \langle\xi\rangle_\gamma^{2l-2t},\\ &\langle\xi\rangle_\gamma^{\pm1} \le \sqrt{2}\langle\eta\rangle_\gamma^{\pm1}(1+|\eta-\xi|^2)^{1/2} \le \sqrt{2}\langle\eta\rangle_\gamma^{\pm1}(1+g_{k,\rho,(x,\xi)}(0,\eta-\xi))^{1/2},\end{aligned}$$

we have

$$\widetilde{K}_k(t,\xi;A,\gamma,l) \le e^{AC_0}\langle\xi\rangle_\gamma^{2l-2t}\langle\eta\rangle_\gamma^{-2l+2t}\,[\![\,\eta\,]\!]_k^{AC_0}\,\widetilde{K}_k(t,\eta;A,\gamma,l). \tag{2.87}$$

From (2.55) we have $[\![\,\eta\,]\!]_k \le C_0(A,l)|\xi-\eta|^{1/\rho}$, where $C_0(A,l)>0$. This, together with (2.87), gives

$$\widetilde{K}_k(t,\xi;A,\gamma,l) \le C_1(A,l)\widetilde{K}_k(t,\eta;A,\gamma,l)(1+g_{k,\rho\,(x,\xi)}(0,\eta-\xi))^{|l-t|+AC_0/(2\rho)},$$

where $C_1(A,l)>0$. □

Define

$$\begin{aligned}\mathcal{E}_k(t;w,A,\gamma,l) =&((D_t-b_k(t,D_x))\psi_\gamma(D_x)w, K_k(D_t-b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\\ &+((w_k(t,D_x)+(\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)},\end{aligned}$$

for $w(t,x)\in C^\infty(\mathbf{R};H^\infty(\mathbf{R}^n_x))$ with $w|_{t\le0}=0$ and $t\in[0,3\delta_1]$, and

$$W_k(t,\xi) = \sum_{\mu=0}^{1} W_{k,\mu}(t,\xi) + \widetilde{W}_{k,2}(t,\xi) \quad \text{for } (t,\xi)\in[0,3\delta_1]\times\mathbf{R}^n,$$

where $K_k = K_k(t,D_x;A,\gamma,l)$. Then we have

$$\begin{aligned}&D_t\mathcal{E}_k(t;w,A,\gamma,l)\\ &=2i\,\mathrm{Im}(\mathrm{Op}((\tau-b_k(t,\xi))^2)\psi_\gamma w, K_k(D_t-b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\\ &+2i\,\mathrm{Re}(\mathrm{Op}(\partial_t b_k(t,\xi))\psi_\gamma w, K_k(D_t-b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\\ &+i((D_t-b_k)\psi_\gamma w, (AW_k(t,D_x)+2(\gamma+\log\langle D_x\rangle_\gamma))K_k(D_t-b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\end{aligned}$$

$$
\begin{aligned}
&- 2i\,\mathrm{Im}((w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&- i(\mathrm{Op}(\partial_t a_k(t,\xi))\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&+ i((w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, (AW_k + 2(\gamma + \log\langle D_x\rangle_\gamma))K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}.
\end{aligned}
$$

From (2.49) – (2.51) we have

$$
\begin{aligned}
&((\tau - b_k(t,\xi))^2 - a_k(t,\xi) + i\partial_t b_k(t,\xi))\psi_\gamma(\xi) \\
&= ((P_k)_{B\Lambda}(t,x,\tau,\xi;R,\varepsilon) - sub\ \sigma(P_k)(t,x,\tau,\xi;R,\varepsilon) \\
&\quad - \tilde{q}_k^0(t,x,\tau,\xi;R,\varepsilon,B))\psi_\gamma(\xi).
\end{aligned}
$$

A simple calculation yields

$$
\begin{aligned}
&D_t\mathcal{E}_k(t;w,A,\gamma,l) = 2i\,\mathrm{Im}((P_k)_{B\Lambda}\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&\quad - 2i\,\mathrm{Im}(sub\ \sigma(P_k)\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&\quad - 2i\,\mathrm{Im}((\tilde{q}_k^0(t,x,D_t,D_x;R,\varepsilon,B) + [\![\, D_x \,]\!]_k^{2\rho_0} + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, \\
&\qquad\qquad K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&\quad + i((D_t - b_k)\psi_\gamma w, (AW_k + 2(\gamma + \log\langle D_x\rangle_\gamma))K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&\quad - i(\mathrm{Op}(\partial_t a_k(t,\xi))\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&\quad + i((w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, (AW_k + 2(\gamma + \log\langle D_x\rangle_\gamma))K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)},
\end{aligned}
$$

where $sub\ \sigma(P_k) = \mathrm{Op}(sub\ \sigma(P_k)(t,x,\tau,\xi;R,\varepsilon))$. Therefore, we have

$$
\begin{aligned}
(2.88)\quad &\partial_t\mathcal{E}_k(t;w,A,\gamma,l) \le \|K_k^{1/2}(P_k)_{B\Lambda}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&+ \|K_k^{1/2}W_{k,1}^{-1/2} sub\ \sigma(P_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&+ \|K_k^{1/2}\tilde{q}_k^0(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&+ \|K_k^{1/2}W_k^{-1/2}\,[\![\, D_x \,]\!]_k^{2\rho_0}\,\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&+ \|K_k^{1/2}(\log\langle D_x\rangle_\gamma)^{3/2}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&- ((D_t - b_k)\psi_\gamma w, ((A-1)W_k + 2\gamma + \log\langle D_x\rangle_\gamma - W_{k,1} - 2) \\
&\qquad\qquad \times K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\
&+ \|K_k^{1/2}W_k^{-1/2}w_k^{-1/2}\,\mathrm{Op}(\partial_t a_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}/2 \\
&- (((A - 1/2)W_k + 2\gamma + 2\log\langle D_x\rangle_\gamma)(w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, \\
&\qquad\qquad K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}.
\end{aligned}
$$

Lemma 2.16. *Let* $\kappa \in \mathbf{R}$, *and let* $q(t,x,\xi;R,\varepsilon) \in S^\kappa_{1,0}([0,3\delta_1] \times T^*\mathbf{R}^n)$ *uniformly in* ε. *Then we have*

$$
(K_k^{1/2}W_{k,1}^{-1/2}) \circ q(t,x,\xi;R,\varepsilon) \circ (K_k^{-1/2}W_{k,1}^{1/2})
$$

$$= q(t,x,\xi;R,\varepsilon) + q_1(t,x,\xi;R,A,\gamma,l,\varepsilon),$$
$$q_1(t,x,\xi;R,A,\gamma,l,\varepsilon)\,[\![\,\xi\,]\!]_k^{\rho} \in S^{\kappa}_{0,0}([0,3\delta_1]\times T^*\mathbf{R}^n) \quad \textit{uniformly in } \gamma \textit{ and } \varepsilon.$$

Moreover, we have

$$K_k^{1/2}\circ q(t,x,\xi;R,\varepsilon)\circ K_k^{-1/2} \in S^{\kappa}_{0,0}([0,3\delta_1]\times T^*\mathbf{R}^n) \quad \textit{uniformly in } \gamma \textit{ and } \varepsilon.$$

Proof. Since $K_k^{1/2}\circ q(t,x,\xi;R,\varepsilon)\circ K_k^{-1/2} = \widetilde{K}_k^{1/2}\circ q(t,x,\xi;R,\varepsilon)\circ \widetilde{K}_k^{-1/2}$, Lemma 2.15 and the results given in §18.5 of [4] prove the lemma. □

By (2.13) and (2.22) we can write

$$\textit{sub } \sigma(P)(t,x,\tau,\xi;R,\varepsilon) = \sum_{\mu=1}^{r_0} \tilde{c}_\mu(t,x)\beta^\mu(t,\tau,\xi) \tag{2.89}$$

for $(t,x,\tau,\xi)\in[0,3\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$, where $\tilde{c}_\mu(t,x)\in C^\infty([0,3\delta_1]\times\mathbf{R}^n)$ ($1\le\mu\le r_0$). So it follows from Lemma 2.4, (2.21), (2.49), (2.51) and (2.89) that

$$\begin{aligned}\textit{sub } \sigma(P_k)(t,x,\tau,\xi;R,\varepsilon) &= q^1_{k,0}(t,x,\xi;R,\varepsilon)(\tau-b_k(t,\xi))\\ &+\sum_{\mu=1}^{r_0}\tilde{c}_\mu(t,x)d_k(t,\xi)\beta^\mu(t,b_k(t,\xi),\xi)/|\xi|^{m-2}\\ &+c_{k,0}(t,x,\xi)a_k(t,\xi)+c_{k,1}(t,\xi)\partial_t a_k(t,\xi)\end{aligned} \tag{2.90}$$

for $(t,x,\xi)\in[0,3\delta_1]\times\mathbf{R}^n\times\overline{\mathcal{C}}$ with $|\xi|\ge 1$, where $d_k(t,\xi)\in S^0_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$. We recall that $c_{k,0}(t,x,\xi), c_{k,1}(t,\xi)\in S^{-1}_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$. By Lemma 2.16 and (2.90) we can also write

$$\begin{aligned} &K_k^{1/2}W_{k,1}^{-1/2}\textit{sub } \sigma(P_k)\psi_\gamma w\\ &= (K_k^{1/2}W_{k,1}^{-1/2}q^1_{k,0}(t,x,D_x;R,\varepsilon)K_k^{-1/2}W_{k,1}^{1/2})K_k^{1/2}W_{k,1}^{-1/2}\\ &\qquad\times(D_t-b_k(t,D_x))\psi_\gamma w\\ &\quad+\sum_{\mu=1}^{r_0}(K_k^{1/2}W_{k,1}^{-1/2}\tilde{c}_\mu(t,x)K_k^{-1/2}W_{k,1}^{1/2})K_k^{1/2}W_{k,1}^{-1/2}d_k(t,D_x)\\ &\qquad\times\mathrm{Op}(\beta^\mu(t,b_k(t,\xi),\xi)/|\xi|^{m-2})\psi_\gamma w\\ &\quad+(K_k^{1/2}W_{k,1}^{-1/2}c_{k,0}(t,x,D_x)K_k^{-1/2}W_{k,1}^{1/2})K_k^{1/2}W_{k,1}^{-1/2}a_k(t,D_x)\psi_\gamma w\\ &\quad+K_k^{1/2}W_{k,1}^{-1/2}c_{k,1}(t,D_x)\,\mathrm{Op}(\partial_t a_k(t,\xi))\psi_\gamma w,\\ &K_k^{1/2}W_{k,1}^{-1/2}\tilde{c}_\mu(t,x)K_k^{-1/2}W_{k,1}^{1/2} = \tilde{c}_\mu(t,x)+\tilde{c}_{\mu,0}(t,x,D_x;A,\gamma,l),\end{aligned} \tag{2.91}$$

where $\tilde{c}_{\mu,0}(t,x,\xi;A,\gamma,l)\,[\![\,\xi\,]\!]_k^\rho \in S^0_{0,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$ uniformly in γ. From Lemma 2.16, the Calderon-Vaillancourt theorem on L^2 boundedness and (2.91) we can see that there are $C_0>0$ and $C(A,l)>0$ satisfying

$$\begin{aligned}
&\|K_k^{1/2}W_{k,1}^{-1/2} sub\ \sigma(P_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\le C(A,l)\Big\{\|K_k^{1/2}W_{k,1}^{-1/2}(D_t-b_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\quad+\sum_{\mu=0}^{r_0}\|K_k^{1/2}W_{k,1}^{-1/2}\,[\![\,D_x\,]\!]_k^{-\rho}\,\mathrm{Op}(\beta^\mu(t,b_k(t,\xi),\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\quad+\sum_{\nu=0}^{1}\|K_k^{1/2}W_{k,1}^{-1/2}\langle D_x\rangle_\gamma^{-1}\,\mathrm{Op}(\partial_t^\nu a_k(t,\xi))\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\Big\} \\
&\quad+C_0\sum_{\mu=1}^{r_0}\|K_k^{1/2}W_{k,1}^{-1/2}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}
\end{aligned}$$

for $t\in[0,3\delta_1]$. It is easy to see that

$$\begin{aligned}
&C(A,l)\|K_k^{1/2}W_{k,1}^{-1/2}\,[\![\,D_x\,]\!]_k^{-\rho}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\le \|K_k^{1/2}W_{k,1}^{-1/2}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\quad+C(A,l)^{1/\rho}\|K_k^{1/2}W_{k,1}^{-1/2}\,[\![\,D_x\,]\!]_k^{-1}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}
\end{aligned}$$

for $t\in[0,3\delta_1]$, since

$$C(A,l)\,[\![\,\xi\,]\!]_k^{-2\rho}\le 1+C(A,l)^{1/\rho}\,[\![\,\xi\,]\!]_k^{-2}\,.$$

From (2.82) we have, with $C'(A,l)>0$,

$$\begin{aligned}
&C(A,l)\|W_{k,1}^{-1/2}\,[\![\,D_x\,]\!]_k^{-\rho}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\le \|W_{k,1}^{-1/2}K_k^{1/2}\,\mathrm{Op}(\beta^\mu(t,b_k,\xi)/|\xi|^{m-2})\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\
&\quad+C'(A,l)\|W_{k,1}^{-1/2}K_k^{1/2}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}
\end{aligned}$$

for $t\in[0,3\delta_1]$. By (2.70) with $\widetilde{\Psi}(\xi)$ replaced by $\psi(\xi)$ there is $C>0$ such that

$$|\langle\xi\rangle_\gamma^{-1}\partial_t^\nu a_k(t,\xi)\psi_\gamma(\xi)|\le C\sqrt{a_k(t,\xi)}\psi_\gamma(\xi)\le Cw_k(t,\xi)^{1/2}\psi_\gamma(\xi)$$

for $\nu=0,1$ and $(t,\xi)\in[0,3\delta_1]\times\mathbf{R}^n$. Noting that

$$|W_{k,1}(t,\xi)^{-1}w_k(t,\xi)^{-1/2}\beta^\mu(t,b_k(t,\xi),\xi)/|\xi|^{m-2}\psi_\gamma(\xi)\le 1$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$, we have

$$\|K_k^{1/2} W_{k,1}^{-1/2} \operatorname{Op}(\beta^\mu(t, b_k, \xi)/|\xi|^{m-2}) \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}$$
$$\le \|K_k^{1/2} W_{k,1}^{1/2} w_k(t, D_x)^{1/2} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}$$

for $t \in [0, 3\delta_1]$, which gives, with $C(A,l) > 0$,

$$\begin{aligned}(2.92) \quad & \|K_k^{1/2} W_{k,1}^{-1/2} sub\ \sigma(P_k) \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \le r_0(C_0+1) \|K_k^{1/2} W_{k,1}^{1/2} w_k^{1/2} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \quad + C(A,l)(\|K_k^{1/2}(D_t - b_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|K_k^{1/2} w_k^{1/2} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)})\end{aligned}$$

for $t \in [0, 3\delta_1]$. From (2.51) and Lemma 2.16 we have

$$\begin{aligned}& K_k^{1/2} \tilde{q}_k^0(t, x, D_t, D_x; R, \varepsilon, B) \\ & = \tilde{q}^0_{k,0}(t, x, D_x; R, A, \gamma, l, \varepsilon, B) \log(1 + \langle D_x \rangle) K_k^{1/2}(D_t - b_k) \\ & \quad + \tilde{q}^0_{k,1}(t, x, D_x; R, A, \gamma, l, \varepsilon, B) \log(1 + \langle D_x \rangle) K_k^{1/2}\end{aligned}$$

where $\tilde{q}^0_{k,\mu}(t, x, D_x; R, A, \gamma, l, \varepsilon, B) \in S^{-1+\mu}_{0,0}(\mathbf{R} \times T^*\mathbf{R}^n)$ uniformly in γ and ε ($\mu = 0, 1$). Therefore, there is $C(A, l, B) > 0$ such that

$$\begin{aligned}(2.93) \quad & \|K_k^{1/2} \tilde{q}_k^0(t, x, D_t, D_x; R, \varepsilon, B) \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \le C(A, l, B) \{ \|K_k^{1/2}(D_t - b_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \\ & \qquad + \|K_k^{1/2} \log\langle D_x \rangle_\gamma \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \}\end{aligned}$$

for $t \in [0, 3\delta_1]$ and $\gamma \ge 2$, since $\log(1 + \langle \xi \rangle) \le \log\langle \xi \rangle_\gamma + \log 2 \le 2 \log\langle \xi \rangle_\gamma$ if $\gamma \ge 2$. Noting that

$$W_k(t,\xi)^{-1/2} [\![\xi]\!]_k^{2\rho_0} \le W_k(t,\xi)^{-1/2} w_k(t,\xi)^{1/2} W_{k,0}(t,\xi) \le W_k^{1/2} w_k(t,\xi)^{1/2},$$

we have

$$(2.94) \quad \|K_k^{1/2} W_k^{-1/2} [\![D_x]\!]_k^{2\rho_0} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \le \|K_k^{1/2} W_k^{1/2} w_k^{1/2} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}$$

for $t \in [0, 3\delta_1]$. Since

$$|W_k(t,\xi)^{-1/2} w_k(t,\xi)^{-1/2} \partial_t a_k(t,\xi) \psi_\gamma(\xi)| \le W_{k,2}(t,\xi)^{1/2} w_k(t,\xi)^{1/2} \psi_\gamma(\xi),$$

we have

$$(2.95) \quad \|K_k^{1/2} W_k^{-1/2} w_k^{-1/2} \operatorname{Op}(\partial_t a_k) \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} \le \|K_k^{1/2} W_k^{1/2} w_k^{1/2} \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}$$

for $t \in [0, 3\delta_1]$. Therefore, it follows from (2.88) and (2.92) – (2.95) that

$$\partial_t \mathcal{E}_k(t; w, A, \gamma, l) \le \|K_k^{1/2}(P_k)_{B\Lambda}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}$$

for $t \in [0, 3\delta_1]$, if $A \ge r_0(C_0+1)+2$ and $\gamma \ge \gamma_k \equiv (C(A,l)+C(A,l,B))/2+1$. Let $A = r_0(C_0+1)+2$ and $\gamma \ge \gamma_k$. Then we have

$$\mathcal{E}_k(t; w, A, \gamma, l) \le \int_0^t \|K_k(s, D_x)^{1/2}(P_k)_{B\Lambda}\psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds$$

for $t \in [0, 3\delta_1]$. Note that

$$\begin{aligned}
&e^{-\gamma t}\langle \xi \rangle_\gamma^{l-\nu_k(A,\delta_1)} \le K_k(t,\xi)^{1/2} \le e^{-\gamma t}\langle \xi \rangle_\gamma^l,\\
&\|D_t\langle D_x\rangle_\gamma^{l-1}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|\langle D_x\rangle_\gamma^l \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\\
&\le C\{\|(D_t - b_k(t,D_x))\langle D_x\rangle_\gamma^{l-1}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|\langle D_x\rangle_\gamma^l \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\}
\end{aligned}$$

for $t \in [0, 3\delta_1]$, where $\nu_k(A, \delta_1) > 0$ and $C > 0$. Then we have

$$\begin{aligned}
(2.96)\qquad &\sum_{\mu=0}^1 \|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l-\mu}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\\
&\le C\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_k(A,\delta_1)}(P_k)_{B\Lambda}\psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds
\end{aligned}$$

for $t \in [0, 3\delta_1]$. Since $(1-\psi_\gamma(\xi))\Psi_\gamma(\xi) = 0$, there is $R_k(t,x,\tau,\xi;R,\varepsilon,B) \in \mathcal{S}_{1,0}^{m-2k+1,-\infty}$ uniformly in γ and ε such that

$$(2.97)\qquad (P_k)_{B\Lambda}\psi_\gamma v^k_{R,\varepsilon} = \psi_\gamma v^{k-1}_{R,\varepsilon} + R_k(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma v.$$

This, together with (2.96), yields

$$\begin{aligned}
(2.98)\qquad &\sum_{\mu=0}^1 \|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l-\mu}\psi_\gamma v^k_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\\
&\le C\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_k(A,\delta_1)}\psi_\gamma v^{k-1}_{R,\varepsilon}|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds\\
&\quad + C_{l,N}(B)\sum_{\mu=0}^{m-2k+1}\int_0^t \|e^{-\gamma s}D_t^\mu\langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds
\end{aligned}$$

for $t \in [0, 3\delta_1]$ and $N \in \mathbf{N}$, where $C, C_{l,N}(B) > 0$. From (2.97) we can write

$$D_t^2\psi_\gamma v^k_{R,\varepsilon} = \psi_\gamma v^{k-1}_{R,\varepsilon} - ((P_k)_{B\Lambda} - D_t^2)\psi_\gamma v^k_{R,\varepsilon} + R_k(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma v.$$

Applying the same argument as for (2.47), we can prove that there are $d^0_{k,\nu,l}(t,x,\tau,\xi;R,\varepsilon,B) \in \mathcal{S}^{1,l-1}_{1,0}$ uniformly in ε, $d^1_{k,\nu,l}(t,x,\tau,\xi;R,\varepsilon,B) \in \mathcal{S}^{\nu-2,l-\nu}_{1,0}$ uniformly in ε and $R_{k,\nu,l}(t,x,\tau,\xi;R,\varepsilon,B,\gamma) \in \mathcal{S}^{m-2k+\nu-1,-\infty}_{1,0}$ uniformly in γ and ε satisfying

$$
\begin{aligned}
D_t^\nu \langle D_x\rangle_\gamma^{l-\nu}\psi_\gamma v^k_{R,\varepsilon} =& d^0_{k,\nu,l}(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma v^k_{R,\varepsilon} \\
&+ d^1_{k,\nu,l}(t,x,D_t,D_x;R,\varepsilon,B)\psi_\gamma v^{k-1}_{R,\varepsilon} \\
&+ R_{k,\nu,l}(t,x,D_t,D_x;R,\varepsilon,B,\gamma)\psi_\gamma v
\end{aligned}
\tag{2.99}
$$

for $\nu \geq 2$. Therefore, it follows from (2.98) and (2.99) that there are positive constants $C_l(B)$ and $C_{l,N}(B)$ ($N \in \mathbf{N}$) such that

$$
\begin{aligned}
&\sum_{\mu=0}^{2k} \|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l-\mu}\psi_\gamma v^k_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)} \\
&\leq C_l(B)\Big\{\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_k(A,\delta_1)}\psi_\gamma v^{k-1}_{R,\varepsilon}|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds \\
&\qquad + \sum_{\mu=0}^{2k-2} \|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l-2-\mu}\psi_\gamma v^{k-1}_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\Big\} \\
&\quad + C_{l,N}(B)\sum_{\mu=0}^{m-1} \|e^{-\gamma t} D_t^\mu \langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v\|^2_{L^2(\mathbf{R}^n_x)}
\end{aligned}
$$

for $t \in [0, 3\delta_1]$ and $N \in \mathbf{N}$. This, together with Lemma 2.7, yields the following

Lemma 2.17. *There are $\gamma_0(B) \geq 1$ and positive constants ν_1, $C_l(B)$ and $C_{l,N}(B)$ ($l \in \mathbf{R}$, $B \geq 1$, $N \in \mathbf{N}$) such that*

$$
\begin{aligned}
&\sum_{\mu=0}^{m} \int_0^t \|e^{-\gamma s} D_t^\mu \langle D_x\rangle_\gamma^{l-\mu} e^{-B\Lambda}\Psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds \\
&\leq C_l(B)\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_1}\psi_\gamma v^0_{R,\varepsilon}|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds \\
&\quad + C_{l,N}(B)\sum_{\mu=0}^{m-1}\int_0^t \|e^{-\gamma s} D_t^\mu \langle D_x\rangle_\gamma^{l-N-\mu}\psi_\gamma v|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds
\end{aligned}
$$

for $B \geq 1$, $l \in \mathbf{R}$, $N \in \mathbf{N}$, $t \in [0, 3\delta_1]$ and $\gamma \geq \gamma_0(B)$.

Remark. $v^0_{R,\varepsilon}(t,x)$ also depends on γ and B (see (2.38)).

2.3. Proof of Theorem 1.2

We can choose $\tilde{\psi}_j(\xi) \in S^0_{1,0}$ ($1 \le j \le N_0$) so that the $\tilde{\psi}_j(\xi)$ are positively homogeneous of degree 0 for $|\xi| \ge 1$, $\operatorname{supp} \tilde{\psi}_j \subset \mathcal{C}_{j,0}$ ($1 \le j \le N_0$) and

$$\sum_{j=1}^{N_0} \tilde{\psi}_j(\xi)^2 (1-\Theta_\gamma(\xi))^2 = (1-\Theta_\gamma(\xi))^2.$$

From (2.35) and (2.38) we have

$$\begin{aligned} (2.100) \quad \psi_{j,\gamma} v^0_{j,R,\varepsilon} &= e^{-B\Lambda_j} \Psi_{j,\gamma} g_{R.\varepsilon} - e^{-B\Lambda_j} \psi_{j,\gamma} R_j(t,x,D_t,D_x;R,\varepsilon,\gamma) \Psi_{j,\gamma} v \\ &\quad + e^{-B\Lambda_j} \psi_{j,\gamma} C_j(t,x,D_t,D_x;R,\varepsilon,\gamma) v \\ &\quad + e^{-B\Lambda_j} \psi_{j,\gamma} ([P_{R,\varepsilon}, \Psi_{j,\gamma}] - C_j(t,x,D_t,D_x;R,\varepsilon,\gamma)) v \\ &= e^{-B\Lambda_j} \Psi_{j,\gamma} g_{R,\varepsilon} - \widetilde{R}_j(t,x,D_t,D_x;R,\varepsilon,\gamma,B)(1-\Theta_\gamma(D_x)) v \\ &\quad + C^1_j(t,x,D_t,D_x;R,\varepsilon,\gamma,B) \langle D_x \rangle_\gamma^{-B} v \\ &\quad + C^2_j(t,x,D_t,D_x;R,\varepsilon,\gamma,B) \Theta_\gamma(D_x) v, \end{aligned}$$

where $\widetilde{R}_j(t,x,\tau,\xi;R,\varepsilon,\gamma,B) \in \mathcal{S}^{m-1,-\infty}_{1,0}$ uniformly in γ and ε, $C^\mu_j(t,x,\tau,\xi;R,\varepsilon,\gamma,B) \in \mathcal{S}^{m-1}_{1,0}$ ($\mu = 1,2$) uniformly in γ and ε and $\psi_{j,\gamma}$ is ψ_γ in §2.2. Indeed, we have

$$e^{-B\Lambda_j(\xi)} = (1+\langle \xi \rangle)^{-B} \le 2^B \langle \xi \rangle_\gamma^{-B}$$

if $C_j(t,x,\tau,\xi;R,\varepsilon,\gamma) \ne 0$ and $|\xi| \ge 3\gamma/2$. Since

$$\tilde{\psi}_j(\xi)(1-\Theta_\gamma(\xi)) e^{-B\Lambda_j(\xi)} \Psi_{j,\gamma}(\xi) = \tilde{\psi}_j(\xi)(1-\Theta_\gamma(\xi)),$$

Lemma 2.17, with (2.100), yields

$$\begin{aligned} (2.101) \quad & \sum_{\mu=0}^m \int_0^t \| e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{l-\mu} (1-\Theta_\gamma(D_x)) v|_{t=s} \|^2_{L^2(\mathbf{R}^n_x)} \, ds \\ & \le C_l(B) \Big\{ \int_0^t \| e^{-\gamma s} \langle D_x \rangle_\gamma^{l+\nu_1} g_{R,\varepsilon}(s,x) \|^2_{L^2(\mathbf{R}^n_x)} \, ds \\ & \quad + \sum_{\mu=0}^{m-1} \int_0^t \| e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{m+l+\nu_1-\mu-1} \Theta_\gamma(D_x) v|_{t=s} \|^2_{L^2(\mathbf{R}^n_x)} \, ds \\ & \quad + \gamma^{-1} \sum_{\mu=0}^{m-1} \int_0^t \| e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{l-\mu} v|_{t=s} \|^2_{L^2(\mathbf{R}^n_x)} \, ds \Big\} \end{aligned}$$

for $B \ge \nu_1 + 1$, $l \in \mathbf{R}$, $t \in [0, 3\delta_1]$ and $\gamma \ge \gamma_0(B)$, with modifications of $C_l(B)$ if necessary. Put $\gamma(l) = \max\{\gamma_0(\nu_1+1), 4C_l(\nu_1+1)\}$, and let $l \in \mathbf{R}$

and $\gamma \geq \gamma(l)$. Then (2.101) gives

$$\begin{aligned}(2.102)\qquad &\sum_{\mu=0}^{m}\int_0^t \|e^{-\gamma s}D_t^{\mu}\langle D_x\rangle_\gamma^{l-\mu}v|_{t=s}\|_{L^2(\mathbf{R}_x^n)}^2\,ds\\ &\leq 2C_l(\nu_1+1)\Big\{\int_0^t \|e^{-\gamma s}\langle D_x\rangle_\gamma^{l+\nu_1}g_{R,\varepsilon}(s,x)\|_{L^2(\mathbf{R}_x^n)}^2\,ds\\ &\quad+\sum_{\mu=0}^{m-1}\int_0^t \|e^{-\gamma s}D_t^{\mu}\langle D_x\rangle_\gamma^{m+l+\nu_1-\mu-1}\Theta_\gamma(D_x)v|_{t=s}\|_{L^2(\mathbf{R}_x^n)}^2\,ds\Big\}\\ &\quad+\sum_{\mu=0}^{m-1}\int_0^t \|e^{-\gamma s}D_t^{\mu}\langle D_x\rangle_\gamma^{l-\mu}\Theta_\gamma(D_x)v|_{t=s}\|_{L^2(\mathbf{R}_x^n)}^2\,ds\end{aligned}$$

for $t \in [0, 3\delta_1]$.

Lemma 2.18. *For $k \in \mathbf{Z}_+$ there is $C_0(k) > 0$ such that*

$$(2.103)\qquad \|D_t^k\langle D_x\rangle_\gamma^{l-k}u\|_{L^2(\mathbf{R}_x^n)} \leq C_0(k)\sum_{\mu=0}^{k}\|(D_t \pm i\gamma)^{\mu}\langle D_x\rangle_\gamma^{l-\mu}u\|_{L^2(\mathbf{R}_x^n)},$$

where $l \in \mathbf{R}$ and $\gamma \geq 1$. Moreover, for $k \in \mathbf{Z}_+$ there is $C(k) > 0$ satisfying

$$\begin{aligned}(2.104)\qquad &C(k)^{-1}\sum_{\mu=0}^{k}\|D_t^{\mu}\langle D_x\rangle_\gamma^{l-\mu}(e^{\pm\gamma t}u)\|_{L^2(\mathbf{R}_x^n)}\\ &\leq \sum_{\mu=0}^{k}\|e^{\pm\gamma t}D_t^{\mu}\langle D_x\rangle_\gamma^{l-\mu}u\|_{L^2(\mathbf{R}_x^n)}\\ &\leq C(k)\sum_{\mu=0}^{k}\|D_t^{\mu}\langle D_x\rangle_\gamma^{l-\mu}(e^{\pm\gamma t}u)\|_{L^2(\mathbf{R}_x^n)},\end{aligned}$$

where $l \in \mathbf{R}$ and $\gamma \geq 1$.

Proof. Noting that

$$D_t^{k+1}\langle D_x\rangle_\gamma^{l-k-1}u = D_t^k\langle D_x\rangle_\gamma^{l-k-1}(D_t \pm i\gamma)u \mp i\gamma D_t^k\langle D_x\rangle_\gamma^{l-k-1}u,$$

we can prove (2.103) by induction on k. (2.104) easily follows from (2.103). □

Since

$$(2.105)\qquad \sum_{\mu=0}^{m}\tau^{2\mu}\langle\xi\rangle_\gamma^{2m-2\mu} \leq \langle(\tau,\xi)\rangle_\gamma^{2m} \leq 2^m\sum_{\mu=0}^{m}\tau^{2\mu}\langle\xi\rangle_\gamma^{2m-2\mu},$$

Lemma 2.18 gives

$$\begin{aligned}(2.106)\qquad &\|\langle (D_t,D_x)\rangle_\gamma^m\langle D_x\rangle_\gamma^l\Theta_\gamma(D_x)(e^{-\gamma t}\chi_0(t)v)\|^2_{L^2(\mathbf{R}^{n+1})}\\ &\geq C^{-1}\sum_{\mu=0}^m\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{m-\mu+l}\Theta_\gamma(D_x)(\chi_0(t)v)\|^2_{L^2(\mathbf{R}^{n+1})}\\ &\geq C^{-1}\sum_{\mu=0}^m\int_0^{3\delta_1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{m-\mu+l}\Theta_\gamma(D_x)v\|^2_{L^2(\mathbf{R}^n_x)}\,dt.\end{aligned}$$

Since $\chi_1(t)=\chi_2(t)=0$ if $t\geq 6\delta_1$, it follows from Lemma 2.5 and (2.104) – (2.106) that

$$\begin{aligned}(2.107)\qquad &\sum_{\mu=0}^m\int_0^{3\delta_1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu}\Theta_\gamma(D_x)v\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\ &\leq C_l'\int_0^{6\delta_1}\|e^{-\gamma t}\langle D_x\rangle_\gamma^{l-m}g_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\ &\quad+\gamma^{-1}C_{l,N}'\sum_{\mu=0}^m\int_0^{6\delta_1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu-N}v\|^2_{L^2(\mathbf{R}^n_x)}\,dt,\end{aligned}$$

where $C_l'>0$ and $C_{l,N}'>0$ ($N\in\mathbf{N}$). From (2.102), (2.107) and Lemma 2.6 we have

$$\begin{aligned}&\sum_{\mu=0}^m\int_0^{6\delta_1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu}v\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\ &\leq C_l''\int_0^{6\delta_1}\|e^{-\gamma t}\langle D_x\rangle_\gamma^{l+\nu_0+\nu_1}g_{R,\varepsilon}\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\ &\quad+\gamma^{-1}C_{l,N}''\sum_{\mu=0}^m\int_0^{6\delta_1}\|e^{-\gamma t}D_t^\mu\langle D_x\rangle_\gamma^{l-\mu-N}v\|^2_{L^2(\mathbf{R}^n_x)}\,dt\end{aligned}$$

for $l\in\mathbf{R}$ and $\gamma\geq\max\{\gamma(l),\gamma(l+\nu_0-1)\}$, where

$$\begin{aligned}C_l'' =&2C_l(\nu_1+1)+(2C_l(\nu_1+1)+1)C_{l+m+\nu_1-1}'\\ &+C(2C_{l+\nu_0-1}(\nu_1+1)+1)(C_{l+m+\nu_0+\nu_1-2}'+1),\\ C_{l,N}'' =&C_{l+m+\nu_0+\nu_1-2,N+m+\nu_0+\nu_1-2}'+(2C_l(\nu_1+1)+1)C_{l+m+\nu_1-1,N+m+\nu_1-1}'.\end{aligned}$$

Therefore, we have the following

Lemma 2.19. *There are $C(l)>0$ ($l\in\mathbf{R}$) and $\tilde\nu>0$ satisfying*

$$\sum_{\mu=0}^m\int_0^{6\delta_1}\|D_t^\mu\langle D_x\rangle^{l-\mu}v\|^2_{L^2(\mathbf{R}^n_x)}\,dt$$

$$\leq C(l) \int_0^{6\delta_1} \|\langle D_x\rangle^{l+\tilde{\nu}} P(t,x,D_t,D_x;R,\varepsilon)v\|^2_{L^2(\mathbf{R}^n_x)}\,dt$$

for $l \in \mathbf{R}$ *and* $v(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n_x))$ *with* $v|_{t\leq 0} = 0$.

Let $f(t,x) \in C^\infty([0,\infty) \times \mathbf{R}^n)$ satisfy $D_t^j f(t,x)|_{t=0} = 0$ ($j \in \mathbf{Z}_+$), and recall that $f_{R,\varepsilon} \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ was defined by (2.23) and $(\mathrm{CP})_{R,\varepsilon}$ has a unique solution $v_{R,\varepsilon}$ in $\mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$, where $R \geq 1$ and $\varepsilon \in (0,1]$. We note that $\operatorname{supp} v_{R,\varepsilon} \subset \{(t,x) \in \mathbf{R}^{n+1};\ (t,x) \in K^+_{(s,y)}$ for some $(s,y) \in \operatorname{supp} f_{R,\varepsilon}\}$, especially, $v_{R,\varepsilon}(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n_x))$. Let $0 < \varepsilon' \leq \varepsilon\,(\leq 1)$, and put $w_{R,\varepsilon,\varepsilon'} = v_{R,\varepsilon} - v_{R,\varepsilon'}$. Then we have

$$\begin{aligned}
&P(t,x,D_t,D_x;R,\varepsilon)w_{R,\varepsilon,\varepsilon'} = f_{R,\varepsilon} - f_{R,\varepsilon'}\\
&\quad + \Theta_{\delta_1}(t) \sum_{j=1}^m \sum_{|\alpha|\leq j-1} (a_{j,\alpha}(t,x;R,\varepsilon') - a_{j,\alpha}(t,x;R,\varepsilon)) D_t^{m-j} D_x^\alpha v_{R,\varepsilon'}\\
&\quad + \frac{i}{2}\Theta_{\delta_1}(t)\Theta_{\delta_1}(-t)(\chi_{R,\varepsilon'}(x) - \chi_{R,\varepsilon}(x))(\partial_t\partial_\tau \hat{p})(t,D_t,D_x)v_{R,\varepsilon'}\\
&\equiv f_{R,\varepsilon,\varepsilon'}
\end{aligned}$$

It follows from (2.11), (2.12), (2.24) and Lemma 2.19 that

$$\begin{aligned}
&\int_0^{6\delta_1} \|\langle D_x\rangle^{l+\tilde{\nu}} f_{R,\varepsilon,\varepsilon'}\|^2_{L^2(\mathbf{R}^n_x)}\,dt \leq \int_0^{4\delta_1} \|\langle D_x\rangle^{l+\tilde{\nu}} (f_{R,\varepsilon} - f_{R,\varepsilon'})\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\
&\quad + C_{R,\varepsilon,\varepsilon'} \sum_{j=0}^m \int_0^{2\delta_1} \|D_t^j \langle D_x\rangle^{l+\tilde{\nu}+m-j} v_{R,\varepsilon'}\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\
&\leq \int_0^{6\delta_1} \|\langle D_x\rangle^{l+\tilde{\nu}} (f_{R,\varepsilon} - f_{R,\varepsilon'})\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\
&\quad + C(l+\tilde{\nu}+m) C_{R,\varepsilon,\varepsilon'} \int_0^{6\delta_1} \|\langle D_x\rangle^{l+2\tilde{\nu}+m} f_{R,\varepsilon'}\|^2_{L^2(\mathbf{R}^n_x)}\,dt\\
&\to 0 \quad \text{as } \varepsilon \downarrow 0,
\end{aligned}$$

where

$$\begin{aligned}
C_{R,\varepsilon,\varepsilon'} = \sup\{&|a_{j,\alpha}(t,x;R,\varepsilon') - a_{j,\alpha}(t,x;R,\varepsilon)| + |\chi_{R,\varepsilon'}(y) - \chi_{R,\varepsilon}(y)|;\\
&t \in [0,2\delta_1],\ x,y \in \mathbf{R}^n,\ 1 \leq j \leq m \text{ and } |\alpha| \leq j-1\}.
\end{aligned}$$

Thus, from Lemma 2.19 we have

$$\sum_{j=0}^m \int_0^{6\delta_1} \|D_t^j \langle D_x\rangle^{l-j} w_{R,\varepsilon,\varepsilon'}\|^2_{L^2(\mathbf{R}^n_x)}\,dt \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

This implies that there is $v_R \in \mathcal{D}'((-\infty, 6\delta_1) \times \mathbf{R}^n)$ such that

$$v_{R,1/j} \to v_R \quad \text{in } \mathcal{D}'((-\infty, 6\delta_1) \times \mathbf{R}^n) \text{ as } j \to \infty,$$

since $v_{R,1/j}(t,x) = 0$ if $t \le 0$. Then we have

$$P(t, x, D_t, D_x; R)v_R = f_R \quad \text{in } \mathcal{D}'((-\infty, 6\delta_1) \times \mathbf{R}^n), \tag{2.108}$$

$$\sum_{j=0}^{m} \int_0^{6\delta_1} \|D_t^j \langle D_x \rangle^{l-j} v_R\|^2_{L^2(\mathbf{R}^n_x)}\, dt \le C_l(f_R) \quad \text{for } l \in \mathbf{R}, \tag{2.109}$$

$$\begin{aligned} \operatorname{supp} v_R \cap (-\infty, 6\delta_1) \times \mathbf{R}^n \subset \{&(t,x) \in [0, 6\delta_1] \times \mathbf{R}^n; \\ &(t,x) \in K^+_{(s,y)} \text{ for some } (s,y) \in \operatorname{supp} f_R\}, \end{aligned} \tag{2.110}$$

where $f_R(t,x) = \Theta_{2\delta_1}(t)\Theta(|x| - R)\tilde{f}(t,x)$ and $C_l(f_R) > 0$. (2.109) gives $v_R \in C^{m-1}([0, 6\delta_1]; H^\infty(\mathbf{R}^n))$. This, together with (2.108), gives $v_R \in C^\infty([0, 6\delta_1]; H^\infty(\mathbf{R}^n))$. Moreover, we have

$$P(t, x, D_t, D_x)v_R = f(t,x) \tag{2.111}$$

for $t \in [0, \delta_1]$ and $x \in \mathbf{R}^n$ with $|x| \le R + 1$. Note that Lemma 2.19 is valid, replacing $P(t, x, D_t, D_x; R, \varepsilon)$ by $P(t, x, D_t, D_x; R)$. Therefore, for $l \in \mathbf{R}$ and $k \ge m$ there is $C_{l,k} > 0$ satisfying

$$\begin{aligned} &\sum_{j=0}^{k} \int_0^{6\delta_1} \|D_t^j \langle D_x \rangle^{l-j} v(t,x)\|^2_{L^2(\mathbf{R}^n_x)}\, dt \\ &\quad \le C_{l,k} \sum_{j=0}^{k-m} \int_0^{6\delta_1} \|D_t^j \langle D_x \rangle^{l+\tilde{\nu}-m-j} P(t, x, D_t, D_x; R)v\|^2_{L^2(\mathbf{R}^n_x)}\, dt \end{aligned} \tag{2.112}$$

for $v(t,x) \in C^\infty([0, 6\delta_1]; H^\infty(\mathbf{R}^n))$ with $D_t^j v(t,x)|_{t \le 0} = 0$ ($j \in \mathbf{Z}_+$). Indeed, for $k \ge m$ there are $d^\nu_{k,\mu}(t, x, \xi; R) \in S^{k-\nu m-\mu}_{1,0}([0, \delta_1] \times T^*\mathbf{R}^n)$ ($\nu = 0, 1$, $0 \le \mu \le m - 1 + \nu(k - 2m + 1)$) such that

$$D_t^m v = \sum_{\mu=0}^{m-1} d^0_{k,\mu}(t, x, D_x; R) D_t^\mu v + \sum_{\mu=0}^{k-m} d^1_{k,\mu}(t, x, D_x; R) D_t^\mu P(t, x, D_t, D_x; R)v,$$

which can be proved by induction on k ($\ge m$).

Lemma 2.20. *Assume that $u \in C^\infty([0, \delta_1] \times \mathbf{R}^n)$, and that $D_t^j u|_{t=0} = 0$ for $j \in \mathbf{Z}_+$. Let $(t_0, x^0) \in [0, \delta_1] \times \mathbf{R}^n$, and assume that*

$$K^-_{(t_0,x^0)} \cap \operatorname{supp} P(t, x, D_t, D_x)u = \emptyset.$$

Then $(t_0, x^0) \notin \operatorname{supp} u$.

Proof. Choose $\tilde{u} \in C^\infty([0,\infty) \times \mathbf{R}^n)$ so that $\tilde{u}|_{t \le \delta_1} = u$ and $\tilde{u}(t,x) = 0$ if $t \ge 2\delta_1$, and put

$$F_R(t,x) = P(t,x,D_t,D_x;R)\tilde{u}(t,x).$$

We choose $R > 0$ so that $K^-_{(t_0,x^0)} \subset \{(t,x) \in [0,\delta_1] \times \mathbf{R}^n;\ |x| \le R\}$. Note that $F_R(t,x) = P(t,x,D_t,D_x)u(t,x)$ if $t \in [0,\delta_1]$ and $|x| \le R+1$. We can write

$$\begin{aligned} &P(t,x,D_t,D_x;R)(\Theta(|x|-R)\tilde{u}(t,x)) = G_R(t,x), \\ &G_R = \Theta(|x|-R)F_R + [P(\cdot;R), \Theta(|x|-R)]\tilde{u} \ \in C^\infty([0,6\delta_1]; H^\infty(\mathbf{R}^n)). \end{aligned}$$

Then there is $w_R \in C^\infty([0,6\delta_1]; H^\infty(\mathbf{R}^n))$ satisfying

$$\begin{cases} P(t,x,D_t,D_x;R)w_R = G_R & \text{in } [0,6\delta_1] \times \mathbf{R}^n, \\ w_R(t,x)|_{t \le 0} = 0. \end{cases}$$

Indeed, putting

$$G_{R,\varepsilon}(t,x) = \Theta_{2\delta_1}(t) \int_{\mathbf{R}^{n+1}} \rho^1_\varepsilon(t-s)\rho_\varepsilon(x-y)G_R(s,y)\,dsdy,$$

we can construct w_R as the limit of $\{w_{R,1/j}\}_{j=1,2,\cdots}$, where $w_{R,\varepsilon}$ is a unique solution in $\mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ of $(\mathrm{CP})_{R,\varepsilon}$ with $f_{R,\varepsilon}$ replaced by $G_{R,\varepsilon}$. From (2.112) we have

$$\Theta(|x|-R)\tilde{u}(t,x) = w_R(t,x) \quad \text{for } (t,x) \in [0,6\delta_1] \times \mathbf{R}^n.$$

It is easy to see that

$$K^-_{(t_0,x^0)} \cap \operatorname{supp} G_R = \emptyset.$$

So (2.110) implies that $(t_0,x^0) \notin \operatorname{supp} u$, since $|x^0| \le R$ and $\Theta(|x|-R) = 1$ near $x = x^0$. □

By (2.111) and Lemma 2.20 we can easily construct a unique solution $u(t,x)$ in $C^\infty([0,\delta_1] \times \mathbf{R}^n)$ satisfying

$$\begin{cases} P(t,x,D_t,D_x)u = f & \text{in } [0,\delta_1] \times \mathbf{R}^n, \\ D_t^j u|_{t=0} = 0 & \text{in } \mathbf{R}^n \ \ (j \in \mathbf{Z}_+), \end{cases}$$

$$\begin{aligned} \text{(2.113)} \quad &\operatorname{supp} u \cap [0,\delta_1] \times \mathbf{R}^n \\ &\subset \{(t,x) \in [0,\delta_1] \times \mathbf{R}^n;\ (t,x) \in K^+_{(s,y)} \text{ for some } (s,y) \in \operatorname{supp} f\}. \end{aligned}$$

Let $u_j(x) \in C^\infty(\mathbf{R}^n)$ ($0 \le j \le m-1$) and $f \in C^\infty([0,\infty) \times \mathbf{R}^n)$. If $u(t,x) \in C^\infty([0,\delta_1] \times \mathbf{R}^n)$ satisfy $P(t,x,D_t,D_x)u(t,x) = f(t,x)$ in $[0,\delta_1] \times \mathbf{R}^n$, then for $\nu \in \mathbf{Z}_+$ there are $a^\nu_{j,\alpha}(t,x) \in C^\infty([0,\infty) \times \mathbf{R}^n)$ ($0 \le j \le m+\nu$, $|\alpha| \le \min\{j,m\}$) such that

$$\begin{aligned} D_t^\nu f(t,x) &= D_t^\nu P(t,x,D_t,D_x)u(t,x) \\ &= D_t^{m+\nu}u(t,x) + \sum_{j=1}^{m+\nu} \sum_{|\alpha|\le\min\{j,m\}} a^\nu_{j,a}(t,x) D_t^{m+\nu-j} D_x^\alpha u(t,x) \end{aligned}$$

for $(t,x) \in [0,\delta_1] \times \mathbf{R}^n$. For $\nu \in \mathbf{Z}_+$ we define inductively

$$u_{m+\nu}(x) = D_t^\nu f(t,x)|_{t=0} - \sum_{j=1}^{m+\nu} \sum_{|\alpha|\le\min\{j,m\}} a^\nu_{j,a}(0,x) D_x^\alpha u_{m+\nu-j}(x).$$

Then, by the Borel theorem there is $U(t,x) \in C^\infty(\mathbf{R}^{n+1})$ satisfying $D_t^j U(t, x)|_{t=0} = u_j(x)$ ($j \in \mathbf{Z}_+$) and $\operatorname{supp} U \subset \mathbf{R} \times \bigcup_{j=0}^\infty \operatorname{supp} u_j$. For $\varepsilon > 0$, putting $u_\varepsilon(t,x) = u(t,x) - \Theta_\varepsilon(t)U(t,x)$ and $f_\varepsilon(t,x) = f(t,x) - P(t,x,D_t,D_x)(u(t,x) - u_\varepsilon(t,x))$, we have

$$D_t^j f_\varepsilon(t,x)|_{t=0} = 0 \quad (j \in \mathbf{Z}_+)$$

$$\begin{cases} P(t,x,D_t,D_x)u_\varepsilon(t,x) = f_\varepsilon(t,x) & \text{in } [0,\delta_1] \times \mathbf{R}^n, \\ D_t^j u_\varepsilon(t,x)|_{t=0} = 0 & \text{in } \mathbf{R}^n \ (j \in \mathbf{Z}_+), \end{cases}$$

since $D_t^\nu P(t,x,D_t,D_x)u(t,x)|_{t=0} = D_t^\nu P(t,x,D_t,D_x)(\Theta_\varepsilon(t)U(t,x))|_{t=0}$ ($\nu \in \mathbf{Z}_+$). Note that

$$\operatorname{supp} f_\varepsilon \subset [0,2\varepsilon] \times \Big(\bigcup_{j=0}^{m-1} \operatorname{supp} u_j \cup \bigcup_{j=0}^{\infty} \operatorname{supp} D_t^j f|_{t=0} \Big) \cup \operatorname{supp} f.$$

Therefore, we can prove that for any $f \in C^\infty([0,\infty) \times \mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)$ ($0 \le j \le m-1$) there is a unique solution $u(t,x)$ in $C^\infty([0,\delta_1] \times \mathbf{R}^n)$ satisfying $(\mathrm{CP})_{\delta_1}$, where $s > 0$ and

$$(\mathrm{CP})_\mathrm{s} \qquad \begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,s] \times \mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1). \end{cases}$$

Let $(t_0,x^0) \in (0,\delta_1] \times \mathbf{R}^n$, and assume that $u_j(x) = 0$ near $\{x \in \mathbf{R}^n;\, (0,x) \in K^-_{(t_0,x^0)}\}$ ($0 \le j \le m-1$) and $f(t,x) = 0$ near $K^-_{(t_0,x^0)} \cap [0,\delta_1] \times \mathbf{R}^n$. Then there is $\varepsilon_0 > 0$ such that $f_\varepsilon(t,x) = 0$ near $K^-_{(t_0,x^0)} \cap [0,\delta_1] \times \mathbf{R}^n$ if $0 < \varepsilon \le \varepsilon_0$. Therefore, (2.113) implies that $(t_0,x^0) \notin \operatorname{supp} u_\varepsilon$ if $0 < \varepsilon \le \varepsilon_0$, which proves that $(t_0,x^0) \notin \operatorname{supp} u$. Put

$$T = \sup\{s \in (0,\infty);\ \text{for any } f \in C^\infty([0,\infty) \times \mathbf{R}^n) \text{ and}$$

$u_j \in C^\infty(\mathbf{R}^n)$ ($0 \le j \le m-1$)
there is a unique solution u in $C^\infty([0,s] \times \mathbf{R}^n)$ of $(\mathrm{CP})_s\}$.

Suppose that $T < \infty$. For $t = T$ we can repeat the same argument as for $t = 0$, and define $\delta_1 > 0$ and $\{\mathcal{C}_j\}$ in the factorization of $p(t, \tau, \xi)$. Then we can show that the Cauchy problem

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) & \text{in } [T - \delta_1/2, T + \delta_1/2] \times \mathbf{R}^n, \\ D_t^j u(t, x)|_{t=T-\delta_1/2} = u_j & \text{in } \mathbf{R}^n \ (\ 0 \le j \le m-1) \end{cases}$$

has a unique solution $u \in C^\infty([T - \delta_1/2, T + \delta_1/2] \times \mathbf{R}^n)$ for any $f \in C^\infty([0,\infty) \times \mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)$ ($0 \le j \le m-1$), which contradicts the definition of T. So we complete the proof of Theorem 1.2.

3. Proof of Theorem 1.3

In this section we assume that the conditions (A-1), (A-2), (H)$'$ and (D) are satisfied. Moreover, we assume that $a_{j,\alpha}(t,x)$ ($0 \le j \le m-1$, $|\alpha| = j, j-1$) are semi-algebraic in $[0,\infty)$ for each $x \in \mathbf{R}^n$ when $n \ge 3$. Let $(t_0, x^0, \xi^0) \in [0,\infty) \times \mathbf{R}^n \times S^{n-1}$ and $\theta_0 > 0$, and let $T(\theta), \Xi_j(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ ($i \le j \le n$) be real-valued functions satisfying the following:

(i) $t_0 + T(\theta) > 0$ for $\theta \in (0, \theta_0]$.

(ii) $T(0) = 0$ and $\Xi(0) = \xi^0$, where $\Xi(\theta) = (\Xi_1(\theta), \cdots, \Xi_n(\theta))$.

(iii) $\Xi(\theta) \in S^{n-1}$ for $\theta \in [0, \theta_0]$.

(iv) $T(\theta)$ and the $\Xi_j(\theta)$ can be expanded into convergent Puiseux series of $\theta \in (0, \theta_0]$.

We say that $T(\theta)$ and $\Xi(\theta)$ satisfy the condition (T, Ξ) if the above conditions (i) – (iv) are satisfied. We can write

$$p(t_0 + T(\theta), \tau, \Xi(\theta)) = \prod_{j=1}^m (\tau - \lambda_j(\theta; T, \Xi)),$$

where $\lambda_j(\theta; T, \Xi) \in C^\infty((0, \theta_0]) \cap C([0, \theta_0])$ ($1 \le j \le m$). We can expand $\lambda_j(\theta; T, \Xi)$ as formal Puiseux series at $\theta = 0$ (see, *e.g.*, [16]). Let $1 \le j_0 \le m$, and put $\tau_0 = \lambda_{j_0}(\theta; T, \Xi)$. Note that $h_{m-1}(t_0, \tau_0, \xi^0) = \prod_{1 \le j \le m, j \ne j_0} (\tau_0 -$

$\lambda_j(0;T,\Xi))^2$. So we may assume that $(\partial_\tau p)(t_0,\tau_0,\xi^0) = 0$. We define the condition $C(t_0,\tau_0,\xi^0,j_0;T,\Xi)$ as follows:

$$\begin{aligned}&\mathrm{Ord}_{\theta\downarrow 0} \min_{s\in\mathcal{R}_0(\Xi(\theta))} |t_0+T(\theta)-s|\,|sub\ \sigma(P)(t_0+T(\theta),x^0,\lambda_{j_0}(\theta;T,\Xi),\Xi(\theta))|\\ &< \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-1}(t_0+T(\theta),\lambda_{j_0}(\theta;T,\Xi),\Xi(\theta))^{1/2}.\end{aligned}$$

Here, for $f\in C([0,\theta_0])$ $\mathrm{Ord}_{\theta\downarrow 0} f = \nu$ ($\in \mathbf{R}$) means that there is $c\in\mathbf{C}\setminus\{0\}$ satisfying $f(\theta) = c\theta^\nu(1+o(1))$ as $\theta\downarrow 0$. If $f(\theta) = O(\theta^N)$ as $\theta \downarrow 0$ for any $N\in\mathbf{Z}_+$, then we define $\mathrm{Ord}_{\theta\downarrow 0} f = \infty$.

Theorem 3.1. *Assume that the condition $C(t_0,\tau_0,\xi^0,j_0;T,\Xi)$ is satisfied. Then the Cauchy problem* (CP) *is not C^∞ well-posed.*

We shall prove Theorem 3.1 in §3.2.

3.1. Preliminaries

Let $(t_0,\xi^0)\in[0,\infty)\times S^{n-1}$, and write

$$\{\tau\in\mathbf{R};\ p(t_0,\tau,\xi^0) = (\partial_\tau p)(t_0,\tau,\xi^0) = 0\} = \{\tau_1,\cdots,\tau_{r(t_0,\xi^0)}\},$$

where $r(t_0,\xi^0)\in\mathbf{Z}_+$ and $\tau_1<\tau_2<\cdots<\tau_{r(t_0,\xi^0)}$. Assume that $r(t_0,\xi^0)\ge 1$. Then, by the Weierstrass preparation theorem there are $\delta_j\equiv\delta_j(t_0,\xi^0)>0$, $\delta_j'\equiv\delta_j'(t_0,\xi^0)>0$, open conic neighborhoods $\Gamma_j\equiv\Gamma_j(t_0,\xi^0)$ of ξ^0, real analytic symbols $e_j(t,\tau,\xi;t_0,\xi^0)$ defined in $\{(t,\tau,\xi)\in[t_0-\delta_j,t_0+\delta_j]\times\mathbf{R}\times(\overline{\Gamma}_j\setminus\{0\});\ \tau_j-\delta_j'\le\tau/|\xi|\le\tau_j+\delta_j'\}$ and real analytic symbols $a_j(t,\xi)$ ($\equiv a_j(t,\xi;t_0,\xi^0)$) and $b_j(t,\xi)$ ($\equiv b_j(t,\xi;t_0,\xi^0)$) defined in $[t_0-\delta_j,t_0+\delta_j]\times(\overline{\Gamma}_j\setminus\{0\})$ ($1\le j\le r(t_0,\xi^0)$) such that the $e_j(t,\tau,\xi;t_0,\xi^0)$ are positively homogeneous of degree $m-2$ in (τ,ξ), $a_j(t,\xi)$ and $b_j(t,\xi)$ ($1\le j\le r(t_0,\xi^0)$) are positively homogeneous of degree 2 and 1 in ξ, respectively, and

$$\begin{aligned}&a_j(t_0,\xi^0)=0,\quad b_j(t_0,\xi^0)=\tau_j,\\ &e_j(t,\tau,\xi;t_0,\xi^0)\neq 0,\\ &p(t,\tau,\xi)=e_j(t,\tau,\xi;t_0,\xi^0)((\tau-b_j(t,\xi))^2-a_j(t,\xi))\end{aligned}$$

for $(t,\tau,\xi)\in[t_0-\delta_j,t_0+\delta_j]\times[\tau_j-\delta_j',\tau_j+\delta_j']\times(\overline{\Gamma}_j\cap S^{n-1})$. If the coefficients of $p(t,\tau,\xi)$ are semi-algebraic in $[0,\infty)$, then $a_j(t,\xi)$ and $b_j(t,\xi)$ ($1\le j\le r(t_0,\xi^0)$) are also semi-algebraic in $[t_0-\delta_j,t_0+\delta_j]\times(\overline{\Gamma}_j\setminus\{0\})$. Indeed, if we write

$$p(t,\tau,\xi)=\prod_{k=1}^m(\tau-\lambda_k(t,\xi)),$$

$$\lambda_1(t,\xi) \le \lambda_2(t,\xi) \le \cdots \le \lambda_m(t,\xi),$$

then the $\lambda_k(t,\xi)$ are semi-algebraic functions. This follows from Theorem 6 in [15] and the fact that the $\lambda_k(t,\xi)$ are real analytic in an open dense semi-algebraic subset of $[0,\infty) \times \mathbf{R}^n$. Therefore, $a_j(t,\xi) \equiv (\lambda_{k(j)}(t,\xi) - \lambda_{k(j)+1}(t,\xi))^2/4$ and $b_j(t,\xi) \equiv (\lambda_{k(j)}(t,\xi) + \lambda_{k(j)+1}(t,\xi))/2$ are semi-algebraic in $[t_0 - \delta_j, t_0 + \delta_j] \times (\overline{\Gamma}_j \setminus \{0\})$, where $k(j)$ satisfies

$$\tau_j = \lambda_{k(j)}(t_0,\xi^0) = \lambda_{k(j)+1}(t_0,\xi^0).$$

Put

$$\begin{aligned}
&\delta_0\,(\equiv \delta_0(t_0,\xi^0)) = \min\{\delta_j;\ 1 \le j \le r(t_0,\xi^0)\},\\
&\delta_0'\,(\equiv \delta_0'(t_0,\xi^0)) = \min\{\delta_j';\ 1 \le j \le r(t_0,\xi^0)\},\\
&\Gamma_0\,(\equiv \Gamma_0(t_0,\xi^0)) = \bigcap_{j=1}^{r(t_0,\xi^0)} \Gamma_j.
\end{aligned}$$

Modifying δ_0 and Γ_0 if necessary, we may assume that

$$|b_j(t,\xi) - \tau_j| + \sqrt{a_j(t,\xi)} < 2\delta_0'/3$$

for $1 \le j \le r(t_0,\xi^0)$ and $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\overline{\Gamma}_0 \cap S^{n-1})$. Putting

$$\lambda_{j,\pm}(t,\xi) = b_j(t,\xi) \pm \sqrt{a_j(t,\xi)},$$

we have

$$\begin{aligned}
&p(t,\lambda_{j,\pm}(t,\xi),\xi) = 0,\\
&|\lambda_{j,\pm}(t,\xi) - \tau_j| < 2\delta_0'/3
\end{aligned}$$

for $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\overline{\Gamma}_0 \cap S^{n-1})$. Moreover, modifying δ_0 and Γ_0 if necessary, we have

$$|p(t,\tau,\xi)| + |\partial_\tau p(t,\tau,\xi)| \ne 0 \tag{3.1}$$

if $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\overline{\Gamma}_0 \cap S^{n-1})$ and $\tau \in \mathbf{R} \setminus \bigcup_{j=1}^{r(t_0,\xi^0)} [\tau_j - 2\delta_0'/3, \tau_j + 2\delta_0'/3]$. When $r(t_0,\xi^0) = 0$, we choose $\delta_0 > 0$ and an open conic neighborhood Γ_0 of ξ^0 so that (3.1) is satisfied, where $\bigcup_{j=1}^{0} \cdots = \emptyset$. Fix $x^0 \in \mathbf{R}^n$. We microlocalize the condition $(\mathrm{L})_0$ as follows:

$(\mathrm{L})_{(t_0,x^0,\xi^0)}$ There is $C>0$ such that

$$\min\Big\{\min_{s\in\mathcal{R}_0(\xi)}|t-s|,1\Big\}|sub\ \sigma(P)(t,x^0,\tau,\xi)|\le Ch_{m-1}(t,\tau,\xi)^{1/2}$$
$$\text{for } (t,\tau,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times\mathbf{R}\times(\Gamma_0\cap S^{n-1}).$$

Lemma 3.2. *The condition* $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ *is equivalent to the following condition:*

$(\mathrm{L})'_{(t_0,x^0,\xi^0)}$ *There is* $C>0$ *such that*

$$\min\Big\{\min_{s\in\mathcal{R}_0(\xi)}|t-s|,1\Big\}|sub\ \sigma(P)(t,x^0,b_j(t,\xi),\xi)|\le C\sqrt{a_j(t,\xi)} \tag{3.2}$$

if $r(t_0,\xi^0)\ge 1$, $1\le j\le r(t_0,\xi^0)$ *and* $(t,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times(\Gamma_0\cap S^{n-1})$.

Proof. If $r(t_0,\xi^0)=0$, then there is $c>0$ satisfying

$$h_{m-1}(t,\tau,\xi)\ge c(1+|\tau|)^{2m-2} \tag{3.3}$$

for $(t,\tau,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times\mathbf{R}\times(\Gamma_0\cap S^{n-1})$. Therefore, $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ always holds if $r(t_0,\xi^0)=0$. So we assume that $r(t_0,\xi^0)\ge 1$. Let $1\le j\le r(t_0,\xi^0)$. Then we can write, with $C>0$,

$$h_{m-1}(t,\lambda_{j,\pm}(t,\xi),\xi)=4a_j(t,\xi)h_j^{\pm}(t,\xi),$$
$$C^{-1}|\xi|^{2m-4}\le h_j^{\pm}(t,\xi)\le C|\xi|^{2m-4}$$

for $(t,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times(\overline{\Gamma}_0\setminus\{0\})$. Therefore, the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ implies that

$(\mathrm{L})''_{(t_0,x^0,\xi^0)}$ There is $C>0$ such that

$$\min\Big\{\min_{s\in\mathcal{R}_0(\xi)}|t-s|,1\Big\}|sub\ \sigma(P)(t,x^0,\lambda_{j,\pm}(t,\xi),\xi)|\le C\sqrt{a_j(t,\xi)}$$

if $1\le j\le r(t_0,\xi^0)$ and $(t,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times(\Gamma_0\cap S^{n-1})$.

Now suppose that $(\mathrm{L})''_{(t_0,x^0,\xi^0)}$ is satisfied. If $(t,\xi)\in[t_0-\delta_0,t_0+\delta_0]\times(\Gamma_0\cap S^{n-1})$ and $\tau\in\mathbf{R}\setminus\bigcup_{j=1}^{r(t_0,\xi^0)}[\tau_j-\delta_0',\tau_j+\delta_0']$, then (3.3) is satisfied with a modification of c if necessary. So we may assume that $1\le j\le r(t_0,\xi^0)$ and $\tau\in[\tau_j-\delta_0',\tau_j+\delta_0']$, Then we have, with $C>0$,

$$|\tau-\lambda_{j,\pm}(t,\xi)|\le Ch_{m-1}(t,\tau,\xi)^{1/2}, \tag{3.4}$$

(3.5) $\sqrt{a_j(t,\xi)} \le \{|\tau - \lambda_{j,+}(t,\xi)| + |\tau - \lambda_{j,-}(t,\xi)|\}/2 \le Ch_{m-1}(t,\tau,\xi)^{1/2}$

for $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\Gamma_0 \cap S^{n-1})$. We can write

$$\begin{aligned} & sub\ \sigma(P)(t,x^0,\tau,\xi) \\ & = sub\ \sigma(P)(t,x^0,\lambda_{j,\pm}(t,\xi),\xi) + \gamma_{j,\pm}(t,\tau,\xi)(\tau - \lambda_{j,\pm}(t,\xi)) \end{aligned}$$

for $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\Gamma_0 \cap S^{n-1})$, where $\gamma_{j,\pm}(t,\tau,\xi)$ are polynomials of τ with coefficients in $C([t_0-\delta_0, t_0+\delta_0] \times (\Gamma_0 \cap S^{n-1}))$. This, together with (3.4) and (3.5), implies that $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ is satisfied. Since there is $C>0$ satisfying

$$\begin{aligned} & |sub\ \sigma(P)(t,x^0,\lambda_{j,\pm}(t,\xi),\xi) - sub\ \sigma(P)(t,x^0,b_j(t,\xi),\xi)| \\ & \le C|\lambda_{j,\pm}(t,\xi) - b_j(t,\xi)| = C\sqrt{a_j(t,\xi)} \end{aligned}$$

if $1 \le j \le r(t_0,\xi^0)$ and $(t,\xi) \in [t_0-\delta_0, t_0+\delta_0] \times (\Gamma_0 \cap S^{n-1})$, the condition $(\mathrm{L})'_{(t_0,x^0,\xi^0)}$ is equivalent to the condition $(\mathrm{L})''_{(t_0,x^0,\xi^0)}$. □

Let $1 \le j \le r(t_0,\xi^0)$, and put

$$\beta_j(t,x,\xi) = sub\ \sigma(P)(t,x,b_j(t,\xi),\xi).$$

Let $\theta_0 > 0$, and let $\Xi_k(\theta)$ ($1 \le k \le n$) be real analytic functions defined on $[0,\theta_0]$ and satisfy $\Xi(0) = \xi^0$, where $\Xi(\theta) = (\Xi_1(\theta), \cdots, \Xi_n(\theta))$. First suppose that $a_j(t_0+t, \Xi(\theta)) \not\equiv 0$ in (t,θ). Define

(3.6) $$\begin{aligned} & \nu_{j,0}\ (\equiv \nu_{j,0}(\Xi)) \\ & = \min\{\nu \in \mathbf{Z}_+;\ \partial_t^l \partial_\theta^\nu a_j(t_0+t,\Xi(\theta))|_{t=0,\,\theta=0} \ne 0 \text{ for some } l \in \mathbf{Z}_+\}. \end{aligned}$$

Then we can write

$$a_j(t_0+t,\Xi(\theta)) = \theta^{\nu_{j,0}} \sum_{k=0}^{\infty} \theta^k A_{j,k}(t) \quad \text{near } \theta = 0. \tag{3.7}$$

Since $A_{j,0}(t) \not\equiv 0$ in t, we put

$$l_j\ (\equiv l_j(\Xi)) = \mathrm{Ord}_{t\downarrow 0}\, A_{j,0}(t)\ (\,<\infty).$$

With a modification of θ_0 if necessary, $\theta^{-\nu_{j,0}} a_j(t_0+t,\Xi(\theta))$ is real analytic in $[-\delta_0,\delta_0] \times [0,\theta_0]$ and

$$\partial_t^l(\theta^{-\nu_{j,0}} a_j(t_0+t,\Xi(\theta)))|_{t=0,\,\theta=0} = 0 \quad \text{if } l < l_j,$$

$$\partial_t^{l_j}(\theta^{-\nu_{j,0}} a_j(t_0+t,\Xi(\theta)))|_{t=0,\theta=0} \neq 0.$$

It follows from the Weierstrass preparation theorem that there are a real analytic function $c_j(t,\theta)$ defined in $[-\delta_0,\delta_0]\times[0,\theta_0]$ and real analytic functions $a_{j,k}(\theta)$ ($1\le k\le l_j$) defined in $[0,\theta_0]$ such that $a_{j,k}(0)=0$ ($1\le k\le l_j$) and

$$c_j(t,\theta)\neq 0, \tag{3.8}$$

$$a_j(t_0+t,\Xi(\theta)) = \theta^{-\nu_{j,0}} c_j(t,\theta)(t^{l_j}+a_{j,1}(\theta)t^{l_j-1}+\cdots+a_{j,l_j}(\theta)) \tag{3.9}$$

for $(t,\theta)\in[-\delta_0,\delta_0]\times[0,\theta_0]$, with modifications of δ_0 and θ_0 if necessary. Write

$$\begin{aligned} &t^{l_j}+a_{j,1}(\theta)t^{l_j-1}+\cdots+a_{j,l_j}(\theta) = \prod_{k=1}^{l_j}(t-t_{j,k}(\theta;\Xi)),\\ &\tau_{j,k}(\theta;\Xi) = \operatorname{Re} t_{j,k}(\theta;\Xi), \end{aligned}$$

where the $t_{j,k}(\theta;\Xi)$ can be expanded into convergent Puiseux series at $\theta=0$. Write

$$\begin{aligned} a_j((t_0+\tau_{j,k}(\theta;\Xi))_+ + t,\Xi(\theta)) &= \theta^{\nu_{j,0}} \sum_{i=0}^{\infty} \theta^i A_{j,i}((t_0+\tau_{j,k}(\theta;\Xi))_+ - t_0 + t)\\ &= \sum_{i=0}^{\infty} A_{j,k,i}(t)\theta^{\nu_{j,0}+i/L}, \end{aligned}$$

where $A_{j,k,0}(t)=A_{j,0}(t)$ and $L\in\mathbf{N}$. Note that $\nu_{j,0}$ is defined as in (3.6). We define

$$\begin{aligned} &\mu_{j,k,i}\,(\equiv \mu_{j,k,i}(\Xi)) = \operatorname{Ord}_{t\downarrow 0} A_{j,k,i}(t),\\ &\Gamma_{0,j,k}(\Xi) = \operatorname{ch}\Big[\bigcup_{i\ge 0,\,\mu_{j,k,i}<\infty}(\{(\nu_{j,0}+i/L,\mu_{j,k,i})\}+(\overline{\mathbf{R}}_+)^2)\Big]. \end{aligned}$$

Here $\operatorname{ch}[A]$ denotes the convex hull of A and $\overline{\mathbf{R}}_+=[0,\infty)$. The $\Gamma_{0,j,k}(\Xi)$ are Newton polygons of $a_j((t_0+\tau_{j,k}(\theta;\Xi))_+ + t,\Xi(\theta))$. Let $1\le k\le l_j$. Suppose that $\beta_j((t_0+\tau_{j,k}(\theta;\Xi))_+ + t, x^0,\Xi(\theta))\not\equiv 0$ in (t,θ). Then we can write

$$t\beta((t_0+\tau_{j,k}(\theta;\Xi))_+ + t, x^0,\Xi(\theta)) = \sum_{i=0}^{\infty} tB_{j,k,i}(t)\theta^{\tilde\nu_{j,k}+i/L},$$

where $\tilde\nu_{j,k}\,(\equiv\tilde\nu_{j,k}(x^0;\Xi))\in\mathbf{Q}\cap[0,\infty)$ and $B_{j,k,0}(t)\not\equiv 0$. Define

$$\tilde\mu_{j,k,i}\,(\equiv\tilde\mu_{j,k,i}(x^0;\Xi)) = 1+\operatorname{Ord}_{t\downarrow 0} B_{j,k,i}(t),$$

$$\Gamma_{1,j,k}(\Xi)\,(\equiv \Gamma_{1,j,k}(x^0;\Xi)) = \mathrm{ch}\Big[\bigcup_{i\ge 0,\,\tilde\mu_{j,k,i}<\infty} (\{(\tilde\nu_{j,k}+i/L,\tilde\mu_{j,k,i})\}+(\overline{\mathbf{R}}_+)^2)\Big].$$

We define $\Gamma_{1,j,k}(\Xi)=\emptyset$ if $\beta_j((t_0+\tau_{j,k}(\theta;\Xi))_+ + t, x^0, \Xi(\theta)) \equiv 0$ in (t,θ). Next suppose that $a_j(t_0+t,\Xi(\theta))\equiv 0$ in (t,θ). Then we define $l_j=1$, $\tau_{j,1}(\theta;\Xi)\equiv 0$ and $\Gamma_{0,j,1}(\Xi)=\emptyset$. We also define $\Gamma_{1,j,1}(\Xi)\,(\equiv \Gamma_{1,j,1}(x^0;\Xi))$ as the Newton polygon of $t\beta_j(t_0+t,x^0,\Xi(\theta))$.

Lemma 3.3. *Let $1\le j\le r(t_0,\xi^0)$. Assume that the following condition* (T) *is satisfied:*

(T) *If $T(\theta)$ is real-valued continuous function defined in $[0,\theta_0]$, $T(\theta)\in C^\infty((0,\theta_0])$, $T(0)=0$, $t_0+T(\theta)>0$ for $\theta\in(0,\theta_0]$ and $T(\theta)$ can be expanded into a formal Puiseux series, then*

$$\begin{aligned}&\mathrm{Ord}_{\theta\downarrow 0}\Big\{\min_{s\in\mathcal{R}_0(\Xi(\theta))} |t_0+T(\theta)-s|\cdot|\beta_j(t_0+T(\theta),x^0,\Xi(\theta))|\Big\}\\ &\ge \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a_j(t_0+T(\theta),\Xi(\theta))}.\end{aligned}$$

Then we have $2\Gamma_{1,j,k}(\Xi)\subset\Gamma_{0,j,k}(\Xi)$ ($1\le k\le l_j$), where $2\Gamma_{1,j,k}(\Xi)=\{(2\nu,2\mu)\in\mathbf{R}^2;\ (\nu,\mu)\in\Gamma_{1,j,k}(\Xi)\}$.

Remark. We can also show that (T) is valid if $2\Gamma_{1,j,k}(\Xi)\subset\Gamma_{0,j,k}(\Xi)$ ($1\le k\le l_j$) (see Lemma 2.2 of [11]).

Proof. We shall repeat the same argument as in the proof of Lemma 2.2 of [11]. Choose real-valued continuous functions $\lambda_k(\theta)$ defined in $[0,\theta_0]$ and subsets I_k of $\{1,2,\cdots,l_j\}$ ($1\le k\le r_j$) so that $\lambda_k(\theta)\in C^\infty((0,\theta_0])$ can be expanded into formal Puiseux series, $\bigcup_{k=1}^{r_j} I_k=\{1,2,\cdots,l_j\}$, $\mathrm{Ord}_{\theta\downarrow 0}((t_0+\tau_{j,k}(\theta;\Xi))_+ - t_0-\lambda_\mu(\theta))=\infty$ for $1\le\mu\le r_j$ and $k\in I_\mu$,

$$\begin{aligned}&\lambda_1(\theta)<\lambda_2(\theta)<\cdots<\lambda_{r_j}(\theta)\quad \text{for } \theta\in(0,\theta_0],\\ &\mathrm{Ord}_{\theta\downarrow 0}(\lambda_{k+1}(\theta)-\lambda_k(\theta))<\infty\quad (\,1\le k\le r_j-1)\end{aligned}$$

and $\lambda_1(\theta)\equiv 0$ if $\mathrm{Ord}_{\theta\downarrow 0}\lambda_1(\theta)=\infty$, where $r_j\in\mathbf{N}$. Let $1\le k\le l_j$ and $p>0$. Putting

$$T_p(t,\theta)=(t_0+\tau_{j,k}(\theta;\Xi))_+ - t_0+\theta^p t\quad (\,1/2\le t\le 1),$$

we have

$$\mathrm{Ord}_{\theta\downarrow 0}\, a_j(t_0+T_p(t,\theta),\Xi(\theta))=\min\{\nu+p\mu;\ (\nu,\mu)\in\Gamma_{0,j,k}(\Xi)\}$$

for a generic $t \in [1/2, 1]$. Moreover, we have

$$\mathrm{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta))} |t_0 + T_p(t, \theta) - s| = p$$

for a generic $t \in [1/2, 1]$. Indeed, we have

$$\begin{aligned} &\mathrm{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta))} |t_0 + T_p(t, \theta) - s| \\ &\geq \mathrm{Ord}_{\theta \downarrow 0} \min_{1 \leq \mu \leq l_j} |(t_0 + \tau_{j,k}(\theta; \Xi))_+ + \theta^p t - (t_0 + \tau_{j,\mu}(\theta; \Xi))_+| = p \end{aligned}$$

for a generic $t \in [1/2, 1]$. On the other hand, we have

$$\begin{aligned} &\mathrm{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta))} |t_0 + T_p(t, \theta) - s| \\ &= \mathrm{Ord}_{\theta \downarrow 0} \min_{s \in \mathcal{R}_0(\Xi(\theta))} |(t_0 + \tau_{j,k}(\theta; \Xi))_+ + \theta^p t - s| \leq p \end{aligned}$$

for a generic $t \in [1/2, 1]$. By assumption we have

$$\begin{aligned} &\mathrm{Ord}_{\theta \downarrow 0} \{\theta^p t \beta_j((t_0 + \tau_{j,k}(\theta; \Xi))_+ + \theta^p t, x^0, \Xi(\theta))\} \\ &\geq \mathrm{Ord}_{\theta \downarrow 0} \sqrt{a_j((t_0 + \tau_{j,k}(\theta; \Xi))_+ + \theta^p t, \Xi(\theta))} \quad \text{for a generic } t \in [1/2, 1]. \end{aligned}$$

This gives

$$\min\{\nu + p\mu;\ (\nu, \mu) \in 2\Gamma_{1,j,k}(\Xi)\} \subset \Gamma_{0,j,k}(\Xi),$$

which proves the lemma. □

3.2. Proof of Theorem 3.1

Let $(t_0, x^0, \xi^0) \in [0, \infty) \times \mathbf{R}^n \times S^{n-1}$, $\theta_0 > 0$ and $1 \leq j_0 \leq m$, and let $T(\theta), \Xi_k(\theta) \in C^\infty((0, \theta_0]) \cap C([0, \theta_0])$ ($1 \leq k \leq n$) be real valued functions satisfy the condition (T, Ξ). Put $\tau_0 = \lambda_{j_0}(0; T, \Xi)$. Assume that the condition $C(t_0, x^0, \xi^0, j_0; T, \Xi)$ is satisfied. It is obvious that

$$p(t_0, \xi^0) = (\partial_\tau p)(t_0, \xi^0) = 0.$$

We use the notation in §3.1. Then there is $j \in \mathbf{N}$ with $1 \leq j \leq r(t_0, \xi^0)$ such that $\tau_0 = \tau_j$. Recall that $a(t_0, \xi^0) = 0$, $b(t_0, \xi^0) = \tau_0$ and

$$p(t, \tau, \xi) = e(t, \tau, \xi)((\tau - b(t, \xi))^2 - a(t, \xi)) \tag{3.10}$$

for $(t, \tau, \xi) \in [t_0 - \delta_0, t_0 + \delta_0] \times [\tau_0 - \delta_0', \tau_0 + \delta_0'] \times (\overline{\Gamma}_0 \cap S^{n-1})$, where $e(t, \tau, \xi) = e_j(t, \tau, \xi; t_0, \xi^0)$, $a(t, \xi) = a_j(t, \xi)$ and $b(t, \xi) = b_j(t, \xi)$. We note that $\lambda_{j_0}(\theta; T, \Xi) = b(t_0 + T(\theta), \Xi(\theta)) \pm \sqrt{a(t_0 + T(\theta), \Xi(\theta))}$. For $\xi \in S^{n-1}$ we define

$$\mathcal{R}(\xi; a) = \begin{cases} \{(\mathrm{Re}\,\lambda)_+;\ \lambda \in \Omega \text{ and } a(\lambda, \xi) = 0\} & \text{if } a(t, \xi) \not\equiv 0 \text{ in } t, \\ \emptyset & \text{if } a(t, \xi) \equiv 0 \text{ in } t. \end{cases}$$

Then we have $\mathcal{R}(\xi; a) \subset \mathcal{R}_0(\xi)$. Indeed, if $a(t, \xi) \not\equiv 0$ in t, then there is a real analytic function $d(t) \not\equiv 0$ satisfying $D_{M-r(\xi)}(t, \xi) = a(t, \xi)d(t)$, where $\xi \in S^{n-1}$ is fixed. Therefore, we have

$$\text{(3.11)} \qquad \mathrm{Ord}_{\theta\downarrow 0} \min_{s \in \mathcal{R}(\Xi(\theta); a)} |t_0 + T(\theta) - s| \le \mathrm{Ord}_{\theta\downarrow 0} \min_{s \in \mathcal{R}_0} |t_0 + T(\theta) - s|.$$

It follows from $C(t_0, x^0, \xi^0, j_0; T, \Xi)$ and (3.11) that

$$\begin{aligned} &\mathrm{Ord}_{\theta\downarrow 0} \min_{s \in \mathcal{R}(\Xi(\theta); a)} |t_0 + T(\theta) - s|\, |sub\ \sigma(P)(t_0 + T(\theta), x^0, \lambda_{j_0}(\theta; T, \Xi), \Xi(\theta))| \\ &< \frac{1}{2} \mathrm{Ord}_{\theta\downarrow 0}\, a(t_0 + T(\theta), \Xi(\theta)). \end{aligned}$$

Now we assume that $a(t, \Xi(\theta)) \not\equiv 0$ in (t, θ). We shall consider the case where $a(t, \Xi(\theta)) \equiv 0$ in (t, θ), later. By (3.7) – (3.9) we can write

$$a(t_0 + t, \Xi(\theta)) = \sum_{k=k_0}^{\infty} a_k(t)\theta^{k/L} = \theta^{k_0/L} c(t, \theta) \prod_{i=1}^{l} (t - t_i(\theta))$$

for $(t, \theta) \in [-\delta_0, \delta_0] \times [0, \theta_0]$, where $L \in \mathbf{N}$, $l = l_j$, $a_{k_0}(t) \not\equiv 0$, $c(t, \theta) \neq 0$ and the $t_i(\theta)$ can be expanded into convergent Puiseux series. We note that $a_k(t)$, $c(t, \theta)$ and $t_i(\theta)$ also depend on Ξ. Put

$$\text{(3.12)} \qquad \begin{aligned} \mu_0 &= \frac{1}{2} \mathrm{Ord}_{\theta\downarrow 0}\, a(t_0 + T(\theta), \Xi(\theta)), \\ \mu_1 &= \mathrm{Ord}_{\theta\downarrow 0} \min_{1 \le i \le l} |t_0 + T(\theta) - (t_0 + \tau_i(\theta))_+| \\ &\qquad\qquad \times |sub\ \sigma(P)(t_0 + T(\theta), x^0, \lambda_{j_0}(\theta; T, \Xi), \Xi(\theta))|, \\ \delta &= \mathrm{Ord}_{\theta\downarrow 0} \min_{1 \le i \le l} |t_0 + T(\theta) - (t_0 + \tau_i(\theta))_+|, \end{aligned}$$

where $\tau_i(\theta) = \mathrm{Re}\, t_i(\theta)$. Since $T(0) = 0$ and $\tau_i(0) = 0$ ($1 \le i \le l$), we have $\delta > 0$. The condition $C(t_0, x^0, \xi^0, j_0; T, \Xi)$ implies that $\mu_1 < \mu_0$. Put

$$T_v(\theta) = T(\theta) + v\theta^{\delta} \quad \text{for } v \in \mathbf{R},$$

and write

$$\text{(3.13)} \qquad sub\ \sigma(P)(t_0 + T_v(\theta), x, \lambda_{j_0}(\theta; T_v, \Xi), \Xi(\theta))$$

$$= \theta^{\mu-\delta}(\hat{c}(v,x) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $\mu \in \mathbf{Q}$, $\hat{c}(v,x) \not\equiv 0$ in (v,x). Then $\hat{c}(v,x)$ is a polynomial of v, whose coefficients are C^∞ functions of x, and $\mu \le \mu_1$. If $\hat{c}(v,x^0) \equiv 0$ in v, we replace $x^0 \in \mathbf{R}^n$ so that $\hat{c}(v,x^0) \not\equiv 0$ in v. There is $c_0 > 0$ satisfying

$$\min_{1\le i\le l} |t_0 + T(\theta) - (t_0 + \tau_i(\theta))_+| \ge c_0\theta^\delta \quad \text{for } \theta \in [0,\theta_0].$$

Since

$$\begin{aligned} &a(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{k_0/L} e(T_v(\theta),\theta) \prod_{i=1}^{l} (T_v(\theta) - t_i(\theta)), \\ &\mathrm{Ord}_{\theta\downarrow 0}(T(\theta) - t_i(\theta)) \le \delta, \end{aligned}$$

$\sqrt{a(t_0 + T_v(\theta), \Xi(\theta))}$ can be expanded into a Puiseux series whose coefficients are real analytic functions of v at $v = 0$. If $\hat{c}(0,x^0) = 0$, we replace $T(\theta)$ and μ_1 by $T(\theta) + v_0\theta^\delta$ and μ, respectively, choosing $v_0 \in (0, c_0/2]$ so that $\hat{c}(v_0,x^0) \neq 0$. Noting that

$$|T(\theta) - \tau_i(\theta)|/2 \le |T(\theta) + v_0\theta^\delta - \tau_i(\theta)| \le 3|T(\theta) - \tau_i(\theta)|/2$$

for $1 \le i \le l$ and $\theta \in [0,\theta_0]$, we have

$$\mu_0 = \mathrm{Ord}_{\theta\downarrow 0} \sqrt{a(t_0 + T_{v_0}(\theta), \Xi(\theta))}.$$

Therefore, we have $\hat{c} \equiv \hat{c}(0,x^0) \neq 0$ and $\mu = \mu_1$ in (3.13) with $v = 0$, and we may assume

$$\begin{aligned} (3.14) \qquad \mu_1 - \delta &= \mathrm{Ord}_{\theta\downarrow 0}\, sub\ \sigma(P)(t_0 + T(\theta), x^0, \lambda_{j_0}(\theta; T, \Xi), \Xi(\theta)) \\ &= \min_{x\in\mathbf{R}^n,\, v\in\mathbf{R}} \mathrm{Ord}_{\theta\downarrow 0}\, sub\ \sigma(P)(t_0 + T_v(\theta), x, \lambda_{j_0}(\theta; T_v, \Xi), \Xi(\theta)). \end{aligned}$$

Let κ and δ' be positive rational constants satisfying $\delta'\kappa < 1$. We shall impose further conditions on κ and δ'. We make an asymptotic change of variables:

$$t = t(s;\rho) \equiv t_0 + T(\rho^{-\kappa}) + \rho^{-\delta\kappa}s, \quad x = x(y;\rho) \equiv x^0 + \rho^{\delta'\kappa-1}y.$$

Put

$$\begin{aligned} &P_\rho(s,y,\sigma,\eta) = P(t(s;\rho), x(y;\rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta), \\ &E(s,y;\rho)\,(\equiv E(s,y;\rho,\varepsilon)) = \exp\Big[i\varepsilon\Big\{\rho^{1-\delta\kappa}\int_0^s \tilde{b}(s_1;\rho)\,ds_1 + \rho^{\delta'\kappa} y\cdot\Xi(\rho^{-\kappa})\Big\}\Big], \end{aligned}$$

where $\varepsilon = \pm 1$ and $\tilde{b}(s;\rho) = b(t(s;\rho), \Xi(\rho^{-\kappa}))$.

Lemma 3.4. *For $k \in \mathbf{N}$ we have*

$$\begin{aligned}&(\rho^{\delta\kappa} D_s)^k E(s, y; \rho)\\&= \{\varepsilon^k \rho^k \tilde{b}(s; \rho)^k + \frac{k(k-1)}{2i} \varepsilon^{k-1} \rho^{k-1} \tilde{b}(s; \rho)^{k-2} (\partial_t b)(t(s; \rho), \Xi(\rho^{-\kappa}))\\&\quad + \Pi_{k-2}(\rho)\} E(s, y; \rho),\end{aligned}$$

where

$$\Pi_\mu(\rho) = \sum_{j=0}^{\mu} \rho^j f_{\mu,j}(t(s; \rho), \Xi(\rho^{-\kappa}))$$

with some C^∞ functions $f_{\mu,j}(t, \xi)$ of (t, ξ) defined near (t_0, ξ^0). Moreover, if $q(\tau, \xi)$ is a homogeneous polynomial of degree m, then we have

$$\begin{aligned}&q(\rho^{\delta\kappa} D_s, \varepsilon\rho\Xi(\rho^{-\kappa})) E(s, 0; \rho)\\&= \{(\varepsilon\rho)^m q(\tilde{b}(s; \rho), \Xi(\rho^{-\kappa}))\\&\quad + \frac{(\varepsilon\rho)^{m-1}}{2i} (\partial_\tau^2 q)(\tilde{b}(s; \rho), \Xi(\rho^{-\kappa}))(\partial_t b)(t(s; \rho), \Xi(\rho^{-\kappa}))\\&\quad + \Pi_{m-2}(\rho)\} E(s, 0; \rho).\end{aligned} \tag{3.15}$$

Proof. The lemma can be proved by induction on k. Then (3.15) is obvious. □

For $(k, \alpha), (\mu, \beta) \in (\mathbf{Z}_+)^{n+1}$ we denote

$$P^{(k,\alpha)}_{(\mu,\beta)}(t, x, \tau, \xi) = D_t^\mu D_x^\beta \partial_\tau^k \partial_\xi^\alpha P(t, x, \tau, \xi).$$

A simple calculation yields

$$\begin{aligned}&E(s, y; \rho)^{-1} P_\rho(s, y, D_s, D_y)(E(s, y; \rho) u(s, y))\\&= E(s, y; \rho)^{-1} \sum_{|(k,\alpha)| \le m} \frac{1}{k!\alpha!} \{P^{(k,\alpha)}(t(s; \rho), x(y; \rho), \rho^{\delta\kappa} D_s, \rho^{1-\delta'\kappa} D_y) E(s, y; \rho)\}\\&\quad \times (\rho^{\delta\kappa} D_s)^k (\rho^{1-\delta'\kappa} D_y)^\alpha u(s, y)\\&= E(s, 0; \rho)^{-1} \sum_{|(k,\alpha)| \le m} \frac{1}{k!\alpha!} \{P^{(k,\alpha)}(t(s; \rho), x(y; \rho), \rho^{\delta\kappa} D_s, \varepsilon\rho\Xi(\rho^{-\kappa})) E(s, 0; \rho)\}\\&\quad \times (\rho^{\delta\kappa} D_s)^k (\rho^{1-\delta'\kappa} D_y)^\alpha u(s, y)\\&= \Big[(\varepsilon\rho)^m p(t(s; \rho), \tilde{b}(s; \rho), \Xi(\rho^{-\kappa}))\\&\quad + (\varepsilon\rho)^{m-1} \Big\{ \frac{1}{2i} (\partial_\tau^2 p)(t(s; \rho), \tilde{b}(s; \rho), \Xi(\rho^{-\kappa}))(\partial_t b)(t(s; \rho), \Xi(\rho^{-\kappa}))\end{aligned}$$

$$
\begin{aligned}
&\qquad + P_{m-1}(t(s;\rho), x(y;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa}))\Big\} + \Pi_{m-2}(\rho) \\
&+ \{(\varepsilon\rho)^{m-1}(\partial_\tau p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) + \Pi_{m-2}(\rho)\}\rho^{\delta\kappa} D_s \\
&+ \Big\{\frac{(\varepsilon\rho)^{m-2}}{2}(\partial_\tau^2 p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) + \Pi_{m-3}(\rho)\Big\}(\rho^{\delta\kappa} D_s)^2 \\
&+ \sum_{k=0}^{2} \sum_{0<|\alpha|\le m-k} \Big\{\frac{(\varepsilon\rho)^{m-k-|\alpha|}}{k!\alpha!} p^{(k,\alpha)}(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\qquad + \Pi_{m-k-|\alpha|-1}(\rho)\Big\}(\rho^{\delta\kappa} D_s)^k (\rho^{1-\delta'\kappa} D_y)^\alpha \\
&+ \sum_{k=3}^{m} \sum_{|\alpha|\le m-k} \Big\{\frac{(\varepsilon\rho)^{m-k-|\alpha|}}{k!\alpha!} p^{(k,\alpha)}(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\qquad + \Pi_{m-k-|\alpha|-1}(\rho)\Big\}(\rho^{\delta\kappa} D_s)^k (\rho^{1-\delta'\kappa} D_y)^\alpha \\
&+ \sum_{i=1}^{m} \sum_{0<k+|\alpha|\le m-i} \Big\{\frac{(\varepsilon\rho)^{m-i-k-|\alpha|}}{k!\alpha!} P_{m-i}^{(k,\alpha)}(t(s;\rho), x(y;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\qquad + \Pi_{m-i-k-|\alpha|-1}(\rho)\Big\}(\rho^{\delta\kappa} D_s)^k (\rho^{1-\delta'\kappa} D_y)^\alpha\Big] u(s,y) \\
&\equiv \varepsilon^m \rho^m \widetilde{P}_\rho(s, y, D_s, D_y) u(s,y),
\end{aligned}
$$

where

$$
\Pi_\mu(\rho) = \sum_{j=0}^{\mu} \rho^j f_{\mu,j}(t(s;\rho), x(y;\rho), \Xi(\rho^{-\kappa}))
$$

with some C^∞ functions $f_{\mu,j}(t,x,\xi)$ of (t,x,ξ) defined near (t_0, x^0, ξ^0). It follows from (2.63), (2.70), (3.10) and (3.12) that

$$
\begin{aligned}
& p(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = -e(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa}))a(t(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = O(\rho^{-2\mu_0\kappa}) \quad \text{as } \rho \to \infty, \\
& (\partial_\tau p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = -(\partial_\tau e)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa}))a(t(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = O(\rho^{-2\mu_0\kappa}) \quad \text{as } \rho \to \infty,
\end{aligned}
$$

$$
\begin{aligned}
\text{(3.16)}\quad & (\partial_\tau^2 p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = 2e(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) + O(\rho^{-2\mu_0\kappa}) \quad \text{as } \rho \to \infty, \\
\text{(3.17)}\quad & \frac{1}{2i}(\partial_\tau^2 p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa}))(\partial_t b)(t(s;\rho), \Xi(\rho^{-\kappa})) \\
&\quad = \frac{i}{2}(\partial_t \partial_\tau p)(t(s;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa}))
\end{aligned}
$$

$$+\frac{i}{2}(\partial_\tau e)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))(\partial_t a)(t(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-2\mu_0\kappa})$$
$$=\frac{i}{2}(\partial_t\partial_\tau p)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-\mu_0\kappa})\quad\text{as }\rho\to\infty,$$
$$(\partial_{\xi_j}p)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))=O(\rho^{-\mu_0\kappa})\quad\text{as }\rho\to\infty,$$

By (3.13), (3.14) and (3.17) we have

$$\begin{aligned}(3.18)\qquad &\frac{1}{2i}(\partial_\tau^2 p)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))(\partial_t b)(t(s;\rho),\Xi(\rho^{-\kappa}))\\ &+P_{m-1}(t(s;\rho),x(y;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))\\ &=sub\ \sigma(P)(t(s;\rho),x(y;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-\mu_0\kappa})\\ &=\rho^{-(\mu_1-\delta)\kappa}(\hat c(s,x^0)+o(1))\quad\text{as }\rho\to\infty,\end{aligned}$$

since $\mu_1<\mu_0$. Noting that

$$|T(\rho^{-\kappa})-\tau_i(\rho^{-\kappa})|/2\le|t(s;\rho)-t_0-\tau_i(\rho^{-\kappa})|\le 3|T(\rho^{-\kappa})-\tau_i(\rho^{-\kappa})|/2$$

if $|s|\le c_0/2$, we choose $s_0\in(0,c_0/2]$ and $\varepsilon=\pm1$ so that

$$(3.19)\qquad \{\varepsilon\hat c(s,x^0)/e(t_0,\tau_0,\xi^0);\ |s|\le s_0\}\cap(-\infty,0]=\emptyset.$$

Assume that

$$1/\kappa>\mu_1+\delta,$$

and put

$$\nu_0=(1-(\mu_1+\delta)\kappa)/2.$$

Then we have

$$2\nu_0+2\delta\kappa-2=-1-(\mu_1-\delta)\kappa.$$

A simple calculation yields

$$\begin{aligned}(3.20)\quad &\exp[-i\rho^{\nu_0}\varphi(s,y;\rho)]\widetilde P_\rho(s,y,D_s,D_y)(\exp[i\rho^{\nu_0}\varphi(s,y;\rho)]u(s,y))\\ &=\Big[\rho^{2\nu_0+2\delta\kappa-2}\{((1/2)(\partial_\tau^2 p)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-1}))\\ &\qquad\times((\partial_s\varphi)^2+(\rho^{-\nu_0}/i)\partial_s^2\varphi+2\rho^{-\nu_0}\partial_s\varphi\cdot D_s+\rho^{-2\nu_0}D_s^2)\\ &\qquad+\varepsilon\rho^{(\mu_1-\delta)\kappa}(sub\ \sigma(P)(t(s;\rho),x(y;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))\\ &\qquad\qquad+O(\rho^{-(\mu_0-\mu_1+\delta)\kappa}))+O(\rho^{-2\nu_0-2\delta\kappa})\}\\ &\quad+\rho^{-2\mu_0\kappa}(\rho^{2\mu_0\kappa}p(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa})))\\ &\quad+\rho^{\nu_0-(2\mu_0-\delta)\kappa-1}(\varepsilon\rho^{2\mu_0\kappa}(\partial_\tau p)(t(s;\rho),\tilde b(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{2\mu_0\kappa-1}))\\ &\qquad\times(\partial_s\varphi+\rho^{-\nu_0}D_s)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{|\alpha|=1} \varepsilon\rho^{\nu_0-\delta'\kappa}\{\rho^{-\mu_0\kappa}(\rho^{\mu_0\kappa}p^{(0,\alpha)}(t(s;\rho),\tilde{b}(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-1})\} \\
&\qquad\qquad \times (\partial_y^\alpha\varphi+\rho^{-\nu_0}D_y^\alpha) \\
&+ \sum_{|\alpha|=2} (\rho^{2\nu_0-2\delta'\kappa}/\alpha!)(p^{(0,\alpha)}(t(s;\rho),\tilde{b}(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-1})) \\
&\qquad \times \{(\nabla_y\varphi)^\alpha+(\rho^{-\nu_0}/i)\partial_y^\alpha\varphi \\
&\qquad\qquad + \rho^{-\nu_0}\sum_{\beta<\alpha,\,|\beta|=1}\partial_y^{\alpha-\beta}\varphi\cdot D_y^\beta+\rho^{-2\nu_0}D_y^\alpha\} \\
&+ \sum_{|\alpha|=1}\rho^{2\nu_0+\delta\kappa-1-\delta'\kappa}(p^{(1,\alpha)}(t(s;\rho),\tilde{b}(s;\rho),\Xi(\rho^{-\kappa}))+O(\rho^{-1})) \\
&\qquad \times \{\partial_s\varphi\cdot\partial_y^\alpha\varphi+(\rho^{-\nu_0}/i)\partial_s\partial_y^\alpha\varphi+\rho^{-\nu_0}\partial_s\varphi\cdot D_y^\alpha \\
&\qquad\qquad + \rho^{-\nu_0}\partial_y^\alpha\varphi\cdot D_s+\rho^{-2\nu_0}D_sD_y^\alpha\} \\
&+ \sum_{3\le k+|\alpha|\le m}\sum_{j=0}^{k}\sum_{\beta\le\alpha}\sum_{l=0}^{k+|\alpha|-j-|\beta|}\rho^{l\nu_0-(1-\delta\kappa)k-\delta'\kappa|\alpha|} \\
&\qquad\qquad\qquad\qquad \times \Phi_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})D_s^jD_y^\beta \\
&+ \sum_{i=1}^{m}\sum_{0<k+|\alpha|\le m-i}\sum_{j=0}^{k}\sum_{\beta\le\alpha}\sum_{l=0}^{k+|\alpha|-j-|\beta|}\rho^{l\nu_0-(1-\delta\kappa)k-\delta'\kappa|\alpha|-i} \\
&\qquad\qquad\qquad\qquad \times \Phi^i_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})D_s^jD_y^\beta\Big]u(s,y)
\end{aligned}
$$

as $\rho\to\infty$, where $\partial_s\varphi=\partial_s\varphi(s,y;\rho)$, $\partial_y^\alpha\varphi=\partial_y^\alpha\varphi(s,y;\rho)$, $\nabla_y\varphi=(\partial_{y_1}\varphi,\cdots,\partial_{y_n}\varphi)$, $I(k,\alpha,j,\beta,l)=\{(h,\gamma)\in(\mathbf{Z}_+)^{n+1};\ h\le k-j,\ |\gamma|\le|\alpha|-|\beta|,\ 1\le h+|\gamma|\le k+|\alpha|-j-|\beta|-l+1\}$ and the $\Phi_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})$ and the $\Phi^i_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})$ denote polynomials of $\{\partial_s^h\partial_y^\gamma\varphi\}_{(h,\gamma)\in I(k,\alpha,j,\beta,l)}$ and ρ^{-1}. We choose $\kappa,\delta'\in\mathbf{Q}$ as follows:

$$
\kappa=(\mu_0+(1+X)\delta)^{-1},\quad \delta'=\mu_0+\delta,
$$

where $X=\min\{1/2,(\mu_0-\mu_1)/(3\delta)\}$. Then we have

$$
\begin{cases}0<\delta'\kappa<1, \quad \nu_0>0, \quad \nu_0+2\delta\kappa-2\ge-2\mu_0\kappa,\\ \nu_0+2\delta\kappa-2>\nu_0-(2\mu_0-\delta)\kappa-1,\end{cases} \tag{3.21}
$$

$$
\nu_0+2\delta\kappa-2\begin{cases}=\nu_0-\delta'\kappa-\mu_0\kappa & \text{if } X=1/2,\\ >\nu_0-\delta'\kappa-\mu_0\kappa & \text{if } X\ne1/2,\end{cases} \tag{3.22}
$$

$$
2\nu_0+2\delta\kappa-2>2\nu_0-2\delta'\kappa, \tag{3.23}
$$

$$(3.24) \qquad 2\nu_0 - 2\delta'\kappa \begin{cases} > \nu_0 + 2\delta\kappa - 2 & \text{if } (\mu_0 - \mu_1)/(3\delta) > 1/2, \\ = \nu_0 + 2\delta\kappa - 2 & \text{if } (\mu_0 - \mu_1)/(3\delta) = 1/2, \\ < \nu_0 + 2\delta\kappa - 2 & \text{if } (\mu_0 - \mu_1)/(3\delta) < 1/2, \end{cases}$$

$$(3.25) \qquad 2\nu_0 + 2\delta\kappa - 2 > 2\nu_0 + \delta\kappa - 1 - \delta'\kappa,$$

$$(3.26) \qquad 2\nu_0 + \delta\kappa - 1 - \delta'\kappa \begin{cases} > \nu_0 + 2\delta\kappa - 2 & \text{if } X > 1/3, \\ = \nu_0 + 2\delta\kappa - 2 & \text{if } X = 1/3, \\ < \nu_0 + 2\delta\kappa - 2 & \text{if } X < 1/3. \end{cases}$$

Moreover, we have

$$(3.27) \qquad \nu_0 + 2\delta\kappa - 2 > k(\nu_0 - (1 - \delta\kappa)) + |\alpha|(\nu_0 - \delta'\kappa) \quad \text{if } k + |\alpha| \geq 3,$$

$$(3.28) \qquad \nu_0 + 2\delta\kappa - 2 > k(\nu_0 - (1 - \delta\kappa)) + |\alpha|(\nu_0 - \delta'\kappa) - i \quad \text{if } i \geq 1 \text{ and } k + |\alpha| > 0.$$

Put

$$\gamma_0 = \delta\kappa(1 - X) \; (\, \geq \delta\kappa/2), \quad l_0 = -[-\nu_0/\gamma_0] - 1.$$

Then we have

$$\begin{aligned} &2\nu_0 + 2\delta\kappa - 2 - (2\nu_0 + \delta\kappa - 1 - \delta'\kappa) = \gamma_0, \\ &2\nu_0 + 2\delta\kappa - 2 - (2\nu_0 - 2\delta'\kappa) = 2\gamma_0, \\ &l_0 = 0 \text{ if and only if } \mu_0 - \mu_1 \leq \delta, \\ &l_0 \geq 1 \text{ if and only if } \mu_0 - \mu_1 > \delta. \end{aligned}$$

We also put

$$\varphi(s, y; \rho) = \sum_{k=0}^{l_0} \rho^{-k\gamma_0} \varphi_k(s, y; \rho) \quad \text{for } (s, y, \rho^{-1}) \in \widetilde{\Omega},$$

where $\widetilde{\Omega} = [-s_0, s_0] \times V_0 \times (0, \rho_0^{-1}]$, $V_0 = \{y \in \mathbf{R}^n;\ |y| \leq 1\}$ and $\rho_0 \gg 1$. By (3.20) – (3.28) we have

$$\begin{aligned} (3.29) \qquad &\exp[-i\rho^{\nu_0}\varphi(s, y; \rho)]\widetilde{P}_\rho(s, y, D_s, D_y)(\exp[i\rho^{\nu_0}\varphi(s, y; \rho)]u(s, y)) \\ &= \rho^{2\nu_0 + 2\delta\kappa - 2}\Big[((1/2)(\partial_\tau^2 p) + O(\rho^{-1})) \\ &\qquad \times ((\partial_s\varphi_0)^2 + \varepsilon((1/2)(\partial_\tau^2 p) + O(\rho^{-1}))^{-1} \\ &\qquad \times \rho^{(\mu_1 - \delta)\kappa}(sub\ \sigma(P)(t(s;\rho), x(y;\rho), \tilde{b}(s;\rho), \Xi(\rho^{-\kappa})) \\ &\qquad\qquad + O(\rho^{-(\mu_0 - \mu_1 + \delta)\kappa})) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{l_0}\rho^{-k\gamma_0}((1/2)(\partial_\tau^2 p)+O(\rho^{-1}))\{2(\partial_s\varphi_0)\cdot(\partial_s\varphi_k)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})\}\\
&+\rho^{-\nu_0}\{((\partial_\tau^2 p)+O(\rho^{-1}))(\partial_s\varphi_0\cdot D_s+\Phi^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{l_0})\\
&\qquad + \rho^{-1/L}\mathcal{L}^\varepsilon(s,y,D_s,D_y;\rho;\varphi_0,\cdots,\varphi_{l_0}))\}\Big]u(s,y)
\end{aligned}
$$

for $(s,y,\rho^{-1})\in\widetilde{\Omega}$, where $L\in\mathbf{N}$, $\partial_s\varphi_k=\partial_s\varphi_k(s,y;\rho)$ and $(\partial_\tau^2 p)=(\partial_\tau^2 p)(t(s;\rho),\tilde{b}(s;\rho),\Xi(\rho^{-\kappa}))$. Here $\Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})$ ($1\le k\le l_0$) and $\Phi^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{l_0})$ are polynomials of derivatives of $\varphi_0(s,y;\rho),\cdots,\varphi_{k-1}(s,y;\rho)$ and $\varphi_0(s,y;\rho),\cdots,\varphi_{l_0}(s,y;\rho)$ with coefficients in $\mathcal{B}(\widetilde{\Omega})$, respectively, and $\mathcal{L}^\varepsilon(s,y,D_s,D_y;\rho;\varphi_0,\cdots,\varphi_{l_0})$ is a differential operator of order m whose coefficients are polynomials of $\{\partial_s^l\partial_y^\alpha\varphi_k(s,y;\rho)\}_{0\le k\le l_0,\,l+|\alpha|\le m}$ with coefficients in $\mathcal{B}(\widetilde{\Omega})$. $\mathcal{B}(\widetilde{\Omega})$ denotes the set of C^∞ functions defined in $\widetilde{\Omega}$ with bounded derivatives. From (3.16), (3.18) and (3.19) we may assume that

$$
\begin{aligned}
\psi(s,y;\rho)\equiv&\varepsilon((1/2)(\partial_\tau^2 p)+O(\rho^{-1}))^{-1}\rho^{(\mu_1-\delta)\kappa}\\
&\times sub\;\sigma(P)(t(s;\rho),x(y;\rho),\tilde{b}(s;\rho),\Xi(\rho^{-\kappa}))\notin(-\infty,0]
\end{aligned}
$$

for $(s,y,\rho^{-1})\in\widetilde{\Omega}$, modifying ρ_0 if necessary, where $\psi(s,y;\rho)$ is the quantity in (3.29). Define

$$
\varphi_0(s,y;\rho)=-i\int_{s_0}^{s}\sqrt{\psi(s_1,y;\rho)}\,ds_1+i|y|^2\quad\text{for }(s,y,\rho^{-1})\in\widetilde{\Omega},
$$

where $\sqrt{z}$ for $z\notin(-\infty,0]$ is the branch satisfying $\operatorname{Re}\sqrt{z}>0$. Then there is $c_1>0$ such that

$$
\begin{aligned}
\operatorname{Im}\varphi_0(s,y;\rho)>c_1(s_0-s)+|y|^2,\\
\partial_s\varphi_0(s,y;\rho)=-i\sqrt{\psi(s,y;\rho)}\neq 0
\end{aligned}
$$

for $(s,y,\rho^{-1})\in\widetilde{\Omega}$. Now we can repeat the argument at the end of §4 of [11] to complete the proof of Theorem 3.1 if $a(t,\Xi(\theta))\not\equiv 0$ in (t,θ). Next consider the case where $a(t,\Xi(\theta))\equiv 0$ in (t,θ). Then we take $T(\theta)\equiv 0$ and $\Xi(\theta)\equiv\xi^0$. Modifying (t_0,x^0,ξ^0) if necessary, we may assume that

$$
sub\;\sigma(P)(t_0,x^0,\lambda_{j_0}(\theta;0,\xi^0),\xi^0)\neq 0,
$$

where $\lambda_{j_0}(\theta;0,\xi^0)\equiv\tau_0$. We make the following asymptotic change of variables:

$$
t=t(s;\rho)=t_0+\rho^{-1/3}s,\quad x=x(y;\rho)=x^0+\rho^{-1/6}y.
$$

Similarly, we have

$$
\begin{aligned}
&\exp[-i\rho^{1/6}\varphi(s,y;\rho)]\widetilde{P}_\rho(s,y,D_s,D_y)(\exp[i\rho^{1/6}\varphi(s,y;\rho)]u(s,y))\\
&= \Big[\rho^{-1}\{((1/2)(\partial_\tau^2 p)(t(s;\rho),\tilde{b}(s;\rho),\xi^0)+O(\rho^{-1}))\\
&\qquad\quad \times((\partial_s\varphi)^2+(\rho^{-1/6}/i)\partial_s^2\varphi+2\rho^{-1/6}\partial_s\varphi\cdot D_s+\rho^{-1/3}D_s^2)\\
&\qquad\quad +\varepsilon\, sub\ \sigma(P)(t(s;\rho),x(y;\rho),\tilde{b}(s;\rho),\xi^0)+O(\rho^{-1})\}\\
&\quad+\sum_{|\alpha|=1}\rho^{-7/6}(p^{(1,\alpha)}(t(s;\rho),\tilde{b}(s;\rho),\xi^0)+O(\rho^{-1}))\\
&\qquad\qquad \times(\partial_s\varphi\cdot\partial_y^\alpha\varphi+(\rho^{-1/6}/i)\partial_s\partial_y^\alpha\varphi+\rho^{-1/6}\partial_s\varphi\cdot D_y^\alpha\\
&\qquad\qquad\qquad +\rho^{-1/6}\partial_y^\alpha\varphi\cdot D_s+\rho^{-1/3}D_sD_y^\alpha)\\
&\quad+\sum_{|\alpha|=2}\sum_{\beta\le\alpha}\sum_{l=0}^{2-|\beta|}\rho^{l/6-5/3}\Phi_{0,\alpha,0,\beta,l}(\varphi,\rho^{-1})D_y^\beta\\
&\quad+\sum_{3\le k+|\alpha|\le m}\sum_{j=0}^{k}\sum_{\beta\le\alpha}\sum_{l=0}^{k+|\alpha|-j-|\beta|}\rho^{l/6-2k/3-5|\alpha|/6}\Phi_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})D_s^jD_y^\beta\\
&\quad+\sum_{i=1}^{m}\sum_{1\le k+|\alpha|\le m-i}\sum_{j=0}^{k}\sum_{\beta\le\alpha}\sum_{l=0}^{k+|\alpha|-j-|\beta|}\rho^{l/6-2k/3-5|\alpha|/6-i}\\
&\qquad\qquad\qquad\qquad \times\Phi^i_{k,\alpha,j,\beta,l}(\varphi,\rho^{-1})D_s^jD_y^\beta\Big]u(s,y),
\end{aligned}
$$

where $\tilde{b}(s;\rho)=b(t(s;\rho),\xi^0)$. Noting that

$$
\begin{aligned}
&1/3-5/3=-4/3\,(<-7/6)\\
&(k+|\alpha|)/6-2k/3-5|\alpha|/6\le -3/2\,(<-7/6)\quad \text{if } k\ge 1 \text{ and } k+|\alpha|\ge 3,\\
&(k+|\alpha|)/6-2k/3-5|\alpha|/6-i\le -3/2\,(<-7/6)\\
&\qquad\qquad\qquad\qquad\qquad \text{if } i\ge 1 \text{ and } k\ge 1 \text{ and } k+|\alpha|\ge 1,
\end{aligned}
$$

we can also repeat the same argument as above, which proves Theorem 3.1.

3.3. Proof of Theorem 1.3

First we assume that $n=2$, and that the Cauchy problem (CP) is C^∞ well-posed. Let $(t_0,x^0,\tau_0,\xi^0)\in[0,\infty)\times\mathbf{R}^2\times\mathbf{R}\times S^1$ satisfy $p(t_0,\tau_0,\xi^0)=(\partial_\tau p)(t_0,\tau_0,\xi^0)=0$. Then there is $j\in\mathbf{N}$ with $1\le j\le r(t_0,\xi^0)$ satisfying $\tau_0=\tau_j$. Here we have used the notations in §3.1. We omit the subscript j, *i.e.*, we write $a(t,\xi)$, $b(t,\xi)$, δ and Γ for $a_j(t,\xi)$, $b_j(t,\xi)$, δ_j and Γ_j, respectively. Moreover, we put

$$\beta(t,x,\xi)=sub\ \sigma(P)(t,x,b(t,\xi),\xi)$$

for $(t,x,\xi) \in [t_0-\delta, t_0+\delta] \times \mathbf{R}^n \times (\overline{\Gamma} \setminus \{0\})$. Let $\boldsymbol{e}$ be a vector in S^1 satisfying $\boldsymbol{e} \perp \xi^0$, and choose $\theta_0 > 0$ so that $\Gamma_0 \equiv \{\lambda(\xi^0 + \theta \boldsymbol{e});\ \lambda > 0 \text{ and } |\theta| \le \theta_0\} \subset \Gamma$. Since $n = 2$, Γ_0 is a conic neighborhood of ξ^0. We put

$$\begin{aligned} &a^{\pm}(t,\theta) = a(t, \xi^0 \pm \theta \boldsymbol{e}), \quad b^{\pm}(t,\theta) = b(t, \xi^0 \pm \theta \boldsymbol{e}), \\ &\beta^{\pm}(t,\theta) = \beta(t, x^0, \xi^0 \pm \theta \boldsymbol{e}). \end{aligned}$$

Suppose that $a^+(t,\theta) \equiv 0$ in (t,θ) and $\beta^+(t,\theta) \not\equiv 0$ in (t,θ). Then, taking $T(\theta) = c\theta$ and $\Xi(\theta) = (\xi^0 + \theta \boldsymbol{e})/|\xi^0 + \theta \boldsymbol{e}|$ with some $c > 0$, we have

$$\begin{aligned} &\mathrm{Ord}_{\theta \downarrow 0} \min\Big\{ \min_{s \in \mathcal{R}_0(\Xi(\theta))} |t_0 + T(\theta) - s|, 1 \Big\} \\ &\quad \times |\beta^+(t_0 + T(\theta), b^+(t_0 + T(\theta), \theta))| \\ &< \mathrm{Ord}_{\theta \downarrow 0} h_{m-1}(t_0 + T(\theta), b^+(t_0 + T(\theta), \theta), \Xi(\theta))^{1/2} = \infty, \end{aligned}$$

since $a^{\pm}(t,\theta) \equiv h_{m-1}(t, b^{\pm}(t,\theta), \Xi(\theta)) \equiv 0$ in (t,θ). Theorem 3.1 implies that (CP) is not C^∞ well-posed, which contradicts the assumption of §3.3. Next suppose that $a^+(t,\theta) \not\equiv 0$ in (t,θ). Then there are $\nu_0, l \in \mathbf{Z}_+$ such that

$$\begin{aligned} &\partial_t^k (\theta^{-\nu_0} a^+(t,\theta))|_{t=t_0,\,\theta=0} = 0 \quad \text{if } k < l, \\ &\partial_t^l (\theta^{-\nu_0} a^+(t,\theta))|_{t=t_0,\,\theta=0} \ne 0. \end{aligned}$$

Therefore, by the Weierstrass preparation theorem there are real analytic functions $e^{\pm}(t,\theta)$ defined in $[t_0-\delta, t_0+\delta] \times [-\theta_0, \theta_0]$ and real analytic functions $a_k^{\pm}(\theta)$ ($1 \le k \le l$) defined in $[-\theta_0, \theta_0]$ such that $a_k^{\pm}(0) = 0$ ($1 \le k \le l$) and

$$a^{\pm}(t,\theta) = \theta^{\nu_0} e^{\pm}(t,\theta) q^{\pm}(t,\theta) \quad \text{for } (t,\theta) \in [t_0 - \delta, t_0 + \delta] \times [-\theta_0, \theta_0],$$

where $q^{\pm}(t,\theta) = (t-t_0)^l + a_1^{\pm}(\theta)(t-t_0)^{l-1} + \cdots + a_l^{\pm}(\theta)$, with modifications of δ and θ_0 if necessary. Now we can repeat the argument in §5 of [11] to prove Theorem 1.3, replacing $b(t,x,\xi)$ and m by $\beta(t,x,\xi)$ and l, respectively, when $n = 2$.

Next assume that $n \ge 3$. Let $(t_0, x^0, \xi^0) \in [0,\infty) \times \mathbf{R}^n \times S^{n-1}$, and assume that $(\mathrm{L})'_{(t_0,x^0,\xi^0)}$ is not satisfied. Then there is $j_0 \in \mathbf{N}$ with $1 \le j_0 \le r(t_0, \xi^0)$ such that (3.2) with $j = j_0$ does not hold. Recall that $b_{j_0}(t_0, \xi^0) = \tau_{j_0}$ and $a_{j_0}(t_0, \xi^0) = 0$. We may assume that $a_{j_0}(t,\xi) \not\equiv 0$ in (t,ξ). Indeed, if $a_{j_0}(t,\xi) \equiv 0$ in (t,ξ), then we have *sub* $\sigma(P)(t_0, x^0, b_{j_0}(t_0,\xi^0), \xi^0) \ne 0$, modifying (t_0, ξ^0) if necessary. With $T(\theta) \equiv 0$ and $\Xi(\theta) \equiv \xi^0$ the condition $C(t_0, x^0, \xi^0, \kappa; T, \Xi)$ is satisfied, where $1 \le \kappa \le m$ and $\lambda_\kappa(t,\xi) = b_{j_0}(t,\xi) - \sqrt{a_{j_0}(t,\xi)} = b_{j_0}(t,\xi)$. Theorem 3.1 implies that the Cauchy problem (CP) is not C^∞ well-posed. We also choose $\delta > 0$ so that

$$(t,\xi) \in (t_0 - \delta_0, t_0 + \delta_0) \times \Gamma_0 \quad \text{if } |t - t_0|^2 + |\xi - \xi^0|^2 \le \delta^2,$$

and define

$$\begin{aligned}
A &= \{(t,\xi,y) \in \mathbf{R}^{n+2};\ |t-t_0|^2 + |\xi-\xi^0|^2 \le \delta^2,\ t \ge 0 \text{ and } y = a_{j_0}(t,\xi)\},\\
B &= \{(t,\xi,y) \in \mathbf{R}^{n+2};\ |t-t_0|^2 + |\xi-\xi^0|^2 \le \delta^2,\ t \ge 0 \text{ and}\\
&\qquad\qquad y = |sub\ \sigma(P)(t,x^0,b_{j_0}(t,\xi),\xi)|^2\},\\
C &= \Big\{(t,\xi,y) \in \mathbf{R}^{n+2};\ |t-t_0|^2 + |\xi-\xi^0|^2 \le \delta^2,\ t \ge 0 \text{ and}\\
&\qquad\qquad y = \min\Big\{\min_{s \in \mathcal{R}_0(\xi/|\xi|)} |t-s|^2, 1\Big\}\Big\}.
\end{aligned}$$

It is obvious that A and B are semi-algebraic sets. Put

$$\begin{aligned}
\Xi_0 &= \{\xi \in \mathbf{R}^n;\ |\xi-\xi^0|^2 \le \delta^2,\ D_M(s_0,\xi) \ne 0 \text{ for some } s_0 \in \mathbf{R}\},\\
\Xi_j &= \{\xi \in \mathbf{R}^n;\ |\xi-\xi^0|^2 \le \delta^2,\ D_{M-j+1}(s,\xi) = 0 \text{ for any } s \in \mathbf{R}\\
&\qquad \text{and } D_{M-j}(s_0,\xi) \ne 0 \text{ for some } s_0 \in \mathbf{R}\} \quad (1 \le j \le M).
\end{aligned}$$

Note that the Ξ_j are semi-algebraic sets and that

$$\Xi_j \cap \Xi_k = \emptyset \quad (j \ne k), \qquad \bigcup_{j=0}^{M} \Xi_j = \{\xi \in \mathbf{R}^n;\ |\xi-\xi^0|^2 \le \delta^2\}.$$

Chosse $\delta' > 0$ so that $\delta' \le 1$ and

$$\{t + i\tau \in \mathbf{C};\ t \in [-\delta', t_0+2],\ \tau \in \mathbf{R},\ |\tau| \le \delta'\} \subset \Omega,$$

where Ω is the complex neighborhood in §1. Put

$$\begin{aligned}
\mathcal{D}_j =& \{(t,\xi) \in \mathbf{R}^{n+1};\ \xi \in \Xi_j,\ D_{M-j}(t_1 + i\tau,\xi) = 0,\ t_1 \in [-\delta', t_0+2],\\
&\tau \in \mathbf{R},\ |\tau| \le \delta',\ t_2 \ge 0,\ t_2^2 = t_1^2 \text{ and } t = (t_1+t_2)/2\} \quad (0 \le j \le M),\\
\mathcal{D} =& \bigcup_{j=0}^{M} \mathcal{D}_j.
\end{aligned}$$

Then we have

$$\begin{aligned}
C =& \{(t,\xi,y) \in \mathbf{R}^{n+2};\ |t-t_0|^2 + |\xi-\xi^0|^2 \le \delta^2,\ t \ge 0\\
&\text{“}(\hat{s},\xi) \in \mathcal{D} \text{ or } \hat{s} = t-1\text{”},\\
&|t-s|^2 \ge |t-\hat{s}|^2 \text{ for any } (s,\xi) \in \mathcal{D} \text{ and } y = |t-\hat{s}|^2\},
\end{aligned}$$

which implies that C is semi-algebraic. Putting

$$\begin{aligned}
\Lambda = \{(\rho,t,\xi,\lambda) \in \mathbf{R}^{n+3};\ &\text{there are } y,u,v,w \in \mathbf{R} \text{ satisfying}\\
&(t,\xi,y) \in A,\ (t,\xi,u) \in B,\ (t,\xi,v) \in C,\ \rho y = 1,\\
&w((|\xi-\xi^0|^2 + |t-t_0|^2)\rho uv + 1) = 1 \text{ and } \lambda = \rho uvw\},
\end{aligned}$$

we can repeat the argument at the end of §6 in [11] to prove Theorem 1.3 when $n \ge 3$.

4. Remarks

Theorem 1.2 is valid for any set-valued function $\mathcal{R}(\xi) : S^{n-1} \ni \xi \mapsto \mathcal{R}(\xi) \in \mathcal{P}(\mathbf{C})$ satisfying (1.2), where $\mathcal{P}(\mathbf{C})$ denotes the power set of $\mathbf{C}$. Therefore, there are various choice in defining the condition (L). The following lemma clarifies the situations.

Lemma 4.1. *The condition* $(\mathrm{L})_0$ *is satisfied if the condition* (L) *is satisfied.*

Lemma 4.1 easily follows from Lemma 4.2 below and the compactness argument. Let U be an open subset of $\mathbf{R}^n$, and let $a(t,\xi)$ be a real analytic function defined in $[0,\delta_0]\times\overline{U}$, where $\delta_0 > 0$. Then there is a compact complex neighborhood Ω_a of $[0,\delta_0]$ such that $a(t,\xi)$ is regarded as an analytic function defined in Ω_a for $\xi \in \overline{U}$. We assume that $a(t,\xi) \ge 0$ for $(t,\xi) \in [0,\delta_0] \times \overline{U}$. Let $b(t,\xi)$ be real analytic in $[0,\delta_0]\times\overline{U}$. Let $\mathcal{R}_U(\xi) : U \ni \xi \mapsto \mathcal{R}_U(\xi) \in \mathcal{P}(\mathbf{C})$ satisfy $\#\mathcal{R}_U(\xi) \le N_U$ for any $\xi \in U$, where $N_U \in \mathbf{N}$. We choose $\delta \in (0,1]$ so that $[-\delta, \delta_0+\delta] \subset \Omega_a$. Let $c \in (0,1]$, and let $\mathcal{R}_{a,\delta,c}(\xi)$ ($\subset \mathbf{C}$) be a set-valued function defined for $\xi \in U$ satisfying the following:

(i) $\sup_{\xi\in U} \#\mathcal{R}_{a,\delta,c}(\xi) < \infty$.

(ii) If $\xi \in U$, $a(t,\xi) \not\equiv 0$ in t, $\lambda \in \Omega_a$, $a(\lambda,\xi) = 0$, $|\operatorname{Im}\lambda| \le \delta$ and $\operatorname{Re}\lambda \in [-\delta, \delta_0+\delta]$, then there is $s \in \mathcal{R}_{a,\delta,c}(\xi)$ satisfying $|\operatorname{Im}\lambda| \ge c|(\operatorname{Re}\lambda)_+ - s|$.

Lemma 4.2. *There are positive constants* δ_1 *and* $A \equiv A(a,\delta,c)$ *independent of* ξ *such that*

$$\min\Big\{\min_{s\in\mathcal{R}_{a,\delta,c}(\xi)} |t-s|, 1\Big\}|b(t,\xi)| \le AC\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in [0,\delta_1] \times U$$

if, with $C \ge 1$,

$$\min\Big\{\min_{s\in\mathcal{R}_U(\xi)} |t-s|, 1\Big\}|b(t,\xi)| \le C\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in [0,\delta_1] \times U,$$

where $\min_{s\in\emptyset} |t-s| = 1$.

Lemma 4.2 can be proved by combining the arguments used in the proof of Lemma 2.1 in [12] and Hironaka's resolution theorem. For details we refer to [17].

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