The Intersection Diagrams of Graphs

Kazumasa Nomura

CONTENTS

PREFACE

The purpose of this thesis is to give a new tool in graph theory. It enables us to analyze the graph structures more systematically. In particular it provides a powerful tool in the theory of distance-regular graphs. We call this method the intersection diagram. We first found this method in research on distance degree regular graphs. Later it turned out to be very useful in the theory of distance-regular graphs.

The concept of distance-regularity of a graph was introduced by N. L. Biggs about twenty years ago. A main subject of the theory of distance-regular graphs is the complete classification of all distance-regular graphs. Since there are only finitely many known distance-regular graphs with given valency k, it might be a natural problem to classify distance-regular graphs with a fixed valency k. However it was rather difficult problem even in the case $k = 3$.

Many, but not all, of the known distance-regular graphs have distance-transitive group actions. A distance-regular graph which have distance-transitive group action is called a distance-transitive graph. Biggs and Smith classified distance-transitive graphs with $k = 3, 4$ ([5], [15], [16], [17]). Recently distance-transitive graphs with $k = 5, 6, 7$ have been classified by Ivanov and Gardinar.

Related to the classification problem, some special types of distance-regular graphs were studied deeply. Bannai, Ito and Damerell completed the classification of Moore graphs ([2], [7]). Egawa and Shrikhande settled the characterization problem of Hamming scheme $H(n, q)$. Jhonson scheme $J(d, n)$ were studied by many authors: Aigner, Bose, Lasker, Moon, etc. Recently Terwilliger found a new method which provides a systematic approach to the characterization problem by using root systems.

In 1983, Ivanov proved an epoch-making theorem which asserts that there are only a finite number of distance-regular graphs with given valency k and girth $g > 3$. By using Ivanov's idea, Biggs, Boshier and ShaweTaylor completed the classification of distance-regular graphs with valencey $k = 3$. Ivanov's proof is not so long but somewhat complicated. The first application of intersection diagrams to distance-regular graphs was obtained when we were searching for a simple proof of Ivanov's theorem. We found a very simple proof of Ivanov's result by using intersection diagrams. In the proof of the classification for $k = 3$, Biggs, Boshier and ShaweTaylor used a purely combinatorial and structure theoretical method as well as Ivanov's method. The proof of their result becomes more clear by using intersection diagrams. After completing the case of valency three, our next problem is to classify distance-regular graphs with valency $k = 4$. We have completed the classification of distance-regular graphs with valency $k = 4$ and girth $g = 3$. Unfortunately, it seems very difficult to classify distance-regular graphs with valency $k = 4$ by our method only. Perhaps it will require both algebraic methods and combinatorial methods.

In Chapter 1, we shall give basic definitions and describe some elementary results concerning the intersection diagrams of general graphs. In Chapter 2, the first application of the intersection diagram will be given. We shall prove some inequalities between distance degrees in distance degree regular graphs. In Chapter 3, we shall discuss intersection diagrams of distance-regular graphs, and give some elementary properties of diagrams and edge patterns. In Chapter 4, we shall give some applications of intersection diagrams to distance-regular graphs. In Section 4.1, we shall give a short proof of Ivanov's Theorem. We have obtained a general inequality between intersection numbers which will be proved in Section 4.2. In section 4.3, we shall prove a result about intersection arrays which is very useful in research on distance-regular graphs. In Chapter 5, we shall describe the classification of distance-regular graphs with valency four and girth three.

I thank E. Bannai, H. Enomoto and T. Ito for their suggestions.

CHAPTER ONE

Preliminaries

1.1. Graphs

In this section we describe some definitions concerning graphs. A *graph* $G = (V, E)$ is a pair of a finite set V and a set E consisting of pairs $\{u, v\}$, $u, v \in V$, $u \neq v$. An element of V is called a *vertex* of G and an element of E is called an *edge* of G. We denote an edge $\{u, v\}$ $(u, v \in V)$ simply by uv. Two vertices u, v are *adjacent* if uv is an edge of G. A *walk* of length r from a vertex u to a vertex v is a series of $(r+1)$ vertices $x_0, x_1, \ldots\ldots, x_r$ of V with $x_i x_{i+1} \in E$ for $0 \leq i < r$. A walk of length r consisting of distinct vertices is called a *path* of length r (or *r-path*). A walk from u to v is said to be *closed* if $u = v$. A *cycle* of length r (or *r-cycle*) is a closed walk of length r $(r \geq 3)$ consisting of r distinct vertices. G is *connected* if for every pair (u, v) of vertices there exists a path from u to v. All graphs will be assumed to be connected. For two vertices u, v, we define the *distance* between u, v to be the length of a shortest path from u to v. Then V becomes a metric space with the metric ∂. The *diameter* $d(G)$ of G is the maximum distance between vertices of G. The *girth* $g(G)$ is the minimum length of cycles in G. For a vertex u in G and for an integer r, the sphere of radius r with the center u is denoted by

$$\Gamma_r(u) = \{ x \in V \mid \partial(x, u) = r \},$$

and the ball of radius r is denoted by

$$\Lambda_r(u) = \{ x \in V \mid \partial(x, u) \leq r \}.$$

The size of $\Gamma_r(u)$ is called the *r-th distance degree* of u. The 1-th distance degree of u is called the *degree* of u and denoted by $d_G(x)$. A complete graph K_n is a graph with n vertices, whose edge set E consists of all pairs of vertices in G. A graph G is said to be *bipartite* if G contains no cycles of odd length. If G is a bipartite graph, there is a partition $V = X \cup Y$, $X \cap Y \neq \emptyset$, and there is no edge inside X and Y. If the edge set E contains all pairs xy ($x \in X, y \in Y$), G is called a *complete bipartite* graph, which is denoted by $K_{m,n}$ where m, n denotes the number of vertices in X, Y respectively.

1.2. Intersection Diagrams

Let $G = (V, E)$ be a connected graph. For two vertices u, v and for two integers r, s, we define

$$D_s^r(u,v) = \Gamma_r(u) \cap \Gamma_s(v),$$

the intersection of two spheres. If there is an edge xy with $x \in D_s^r(u,v)$, $y \in D_{s'}^{r'}(u,v)$, then we get

$$r' = \partial(u,y) \leq \partial(u,x) + \partial(x,y) = r + 1,$$

$$r = \partial(u,x) \leq \partial(u,y) + \partial(y,x) = r' + 1.$$

Similarly we have $s' \leq s+1$, $s \leq s'+1$. So we get the following lemma.

Lemma 1.1. *If* $|r-r'| \geq 2$ *or* $|s-s'| \geq 2$, *there is no edge between* $D_s^r(u,v)$ *and* $D_{s'}^{r'}(u,v)$.

Let $\partial(u,v) = t$, and take a vertex x in $D^{\tau}_{s}(u,v)$. Then we have

$$\tau = \partial(u,x) \leq \partial(u,v) + \partial(v,x) = t + s,$$

$$s = \partial(v,x) \leq \partial(v,u) + \partial(u,x) = t + \tau.$$

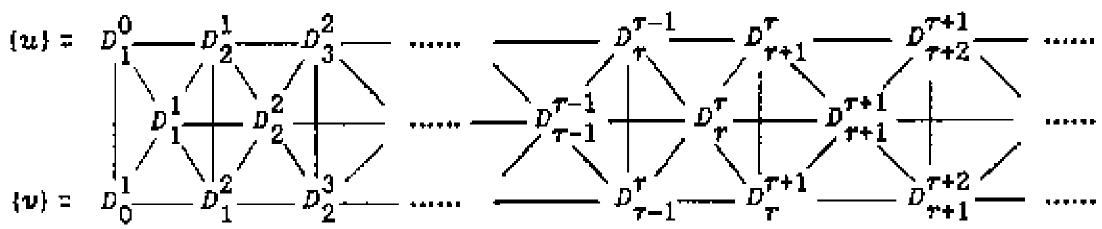
This implies $|\tau - s| \leq t$, thus we get

Lemma 1.2. *If* $|\tau - s| > \partial(u,v)$, $D^{\tau}_{s}(u,v)$ *is empty.*

For fixed vertices u, v in G, we call the family $\{D^{\tau}_{s}(u,v)\}_{\tau,s}$ the *intersection diagram* of G with respect to (u,v). Now fix an edge uv in G, and put $D^{\tau}_{s} = D^{\tau}_{s}(u,v)$. By lemma 1.2, D^{τ}_{s} is empty for $|\tau - s| \geq 2$, so we have

$$\{D^{\tau}_{s}\}_{\tau,s} = \{D^{\tau}_{\tau+1}\}_{\tau} \cup \{D^{\tau}_{\tau}\}_{\tau} \cup \{D^{\tau+1}_{\tau}\}_{\tau} \quad \text{(disjoint).}$$

We draw the intersetion diagram as follows.



where a line between two components of the family $\{D^{\tau}_{s}\}_{\tau,s}$ indicates possibility of existence of edges connecting between them.

1.3. Edge Patterns

Let $G = (V, E)$ be a connected graph and take two vertices u, v of G. For a vertex x in $D^{\tau}_{s}(u,v)$, we put the number of edges from x to a component $D^{\tau'}_{s'}(u,v)$ as follows.

$$e^{\tau'-\tau}_{s'-s}(x;\, u,\, v) = |\, \Gamma_1(x) \cap D^{\tau'}_{s'}(u,v)\,|.$$

We call the family $\{e^{\mu}_{\nu}(x;\, u,\, v)\}_{\mu,\nu}$ the *edge patterns* of a vertex x in the intersection diagram of G with respect to (u,v). We fix u, v, and put $e^{\mu}_{\nu}(x) = e^{\mu}_{\nu}(x;\, u,\, v)$ for every integer μ, ν. By Lemma 1.1, $e^{\mu}_{\nu}(x) = 0$ if $|\mu-\nu| < 1$. Thus we may only consider the followings.

$$e^{+1}_{+1}(x),\ e^{+1}_{0}(x),\ e^{+1}_{-1}(x),\ e^{0}_{+1}(x),\ e^{0}_{0}(x),\ e^{0}_{-1}(x),\ e^{-1}_{+1}(x),\ e^{-1}_{0}(x),\ e^{-1}_{-1}(x).$$

Thus we get the following lemma.

Lemma 1.3. *For $x \in D^{\tau}_{s}(u,v)$ we have the following equalities where μ and ν ranges over* $\{-1, 0, +1\}$.

(i) $\displaystyle\sum_{\mu,\nu} e^{\mu}_{\nu}(x) = d_G(x),$

(ii) $\displaystyle\sum_{\nu} e^{+1}_{\nu}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau+1}(u)\,|,$

$\displaystyle\sum_{\nu} e^{0}_{\nu}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau}(u)\,|,$

$\displaystyle\sum_{\nu} e^{+1}_{\nu}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau}(u)\,|,$

(iii) $\displaystyle\sum_{\mu} e^{\mu}_{+1}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau+1}(v)\,|,$

$\displaystyle\sum_{\mu} e^{\mu}_{0}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau}(v)\,|,$

$\displaystyle\sum_{\mu} e^{\mu}_{-1}(x) = |\, \Gamma_1(x) \cap \Gamma_{\tau-1}(v)\,|.$

CHAPTER TWO

Distance Degree Regular Graphs

2.1. An Inequality on Distance Degrees

Let $G = (V, E)$ be a connected graph with the vertex set V and the edge set E. G is said to be *distance degree regular* if the relation

$$|\Gamma_i(u)| = |\Gamma_i(v)|$$

holds for any vertices u, v and nonnegative integer i. In this case, the i-th distance degree of a distance degree regular graph G is the number $|\Gamma_i(u)|$, which will be denoted by $k_i(G)$ or simply by k_i. We remark that if G is distance degree regular, then

$$|\Lambda_i(u)| = \sum_{j=0}^{i} k_j \tag{1}$$

holds. We shall show

Theorem 2.1. *Let G be a connected and distance degree regular graph and $d(G) \geq 2$. Then, for every integer r, $1 \leq r < d(G)$, we have*

$$3k_r(G) \geq 2(k_1(G) + 1).$$

Theorem 2.2. *Let G be a connected and distance degree regular graph and $d(G) \geq 2$. If*

$$3k_r(G) = 2(k_1(G) + 1) \tag{2}$$

holds for some integer r, $1 \leq r < d(G)$, we have

G isomorphic to $C_n[K_m]$,

where $n = 2d(G) + 1$ *or* $2d(G)$ *and* $m = k_r(G) / 2$.

In the above theorem, $C_n[K_m]$ denotes the composition of C_n by K_m, where C_n is the n-cycle and K_m is the complete graph of m vertices. The *composition* $G = G_1[G_2]$ of $G_1 = (V_1, E_1)$ by $G_2 = (V_2, E_2)$ is a graph with the vertex set $V_1 \times V_2$, and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are defined to be adjacent if $[u_1v_1 \in E_1]$ or $[u_1 = v_1$ and $u_2v_2 \in E_2]$.

To prove the above theorems, we shall use the following simple lemma.

Lemma 2.3. *Let* a, b *and* c *be vertices of* G *such that* $\partial(a,b) = n$, $\partial(b,c) = m$ *and* $\partial(a,c) = n + m$, *then we have*

$$\Lambda_m(a) \cup \Lambda_n(c) \subset \Lambda_{m+n}(b).$$

In particular

$$|\Lambda_{m+n}(b)| \geq |\Lambda_m(a)| + |\Lambda_n(c)| - |\Lambda_m(a) \cap \Lambda_n(c)| \tag{3}$$

The theorems are obvious for $r = 1$, so we assume that $G = (V, E)$ is connected and distance degree regular and $d(G) > 2$ and $1 < r < d(G)$ in the rest of chapter 2.

2.2. Proof of Theorem 2.1

For every edge $uv \in E$ and positive integers i and j, the following hold:

$$\text{If } |i - j| \geq 2 \text{ then } \Gamma_i(u) \cap \Gamma_j(v) = \emptyset. \tag{4}$$

$$|\Gamma_{i+1}(u) \cap \Gamma_i(v)| + |\Gamma_i(u) \cap \Gamma_i(v)| + |\Gamma_{i-1}(u) \cap \Gamma_i(v)| = k_i. \qquad (5)$$

Since

$$|\Gamma_1(u) \cap \Gamma_0(v)| = |\Gamma_0(u) \cap \Gamma_1(v)| = 1,$$

it follows from (5) by induction on i that

$$|\Gamma_{i+1}(u) \cap \Gamma_i(v)| = |\Gamma_i(u) \cap \Gamma_{i+1}(v)|. \qquad (6)$$

By (1) and (3), we also get, if $\partial(u,v) = r$, that

$$|\Lambda_{r-1} \cap \Lambda_1(v)| = |\Gamma_{r-1}(u) \cap \Gamma_1(v)| \geq 1 + k_1 - k_r. \qquad (7)$$

Now we choose two vertices u and z such that $\partial(u,z) = r + 1$. Let $(u, v, w, \dots, z)$ be one of the shortest paths from u to z. By (7),

$$|\Gamma_{r-1}(v) \cap \Gamma_1(z)| \geq 1 + k_1 - k_r.$$

Since

$$\Gamma_{r-1}(v) \cap \Gamma_1(z) \subset \Gamma_r(u) \cap \Gamma_{r-1}(v), \qquad (8)$$

we have

$$|\Gamma_r(u) \cap \Gamma_{r-1}(v)| \geq 1 + k_1 - k_r. \qquad (9)$$

We also have

$$|\Gamma_{r+1}(u) \cap \Gamma_r(v)| + |\Gamma_r(u) \cap \Gamma_r(v)| \geq 1 + k_1 - k_r. \qquad (10)$$

To prove the above inequality (10), we consider three cases.

Case (*i*). There exists a vertex $x \in \Gamma_1(z) \cap \Gamma_{r+2}(u) \cap \Gamma_{r+1}(v)$.

Since $\partial(w,x) = r$, (7), and

$$\Gamma_{r-1}(w) \cap \Gamma_1(x) \subset \Gamma_{r+1}(u) \cap \Gamma_r(v),$$

we have

$$|\Gamma_{r+1}(u) \cap \Gamma_r(v)| \geq 1 + k_1 - k_r.$$

Case (*ii*). There exists a vertex $x \in \Gamma_1(z) \cap \Gamma_{r+1}(u) \cap \Gamma_{r+1}(v)$.

Since $\partial(w,x) = r$, we have

$$|\Gamma_{r-1}(w) \cap \Gamma_1(x)| \geq 1 + k_1 - k_r,$$

by (7). We also get

$$\Gamma_{r-1}(w) \cap \Gamma_1(x) \subset (\Gamma_{r+1}(u) \cap \Gamma_r(v)) \cup (\Gamma_r(u) \cap \Gamma_r(v))$$

for this case. Hence (10) holds again.

Case (*iii*). There exists no vertex in

$$\Gamma_1(x) \cap (\Gamma_{r+2}(u) \cup \Gamma_{r+1}(u)) \cap \Gamma_{r+1}(v).$$

In this case we have

$$\Lambda_1(x) = \left[\bigcup_{i=r}^{r+2} \Gamma_i(u)\right] \cap \left[\bigcup_{j=r-1}^{r+1} \Gamma_j(v)\right] \cap \Lambda_1(x)$$

$$\subset (\Gamma_{r+1}(u) \cap \Gamma_r(v)) \cup \Gamma_r(u).$$

Hence

$$1 + k_1 \leq |\Gamma_{r+1}(u) \cap \Gamma_r(v)| + k_r.$$

Theorem 2.1 follows from (5), (9) and (10).

2.3. Proof of Theorem 2.2

Let u and x be any two vertices such that $\partial(u,x) = r+1$ and $(u, v, w, \ldots, x)$ be one of the shortest paths from u to x. Condition (2) implies that

$$1 + k_1 - k_r = k_r / 2,$$

and forces equality in (8) and (9), so we have

$$\Lambda_{r-1}(v) \cap \Lambda_1(x) = \Gamma_r(u) \cap \Gamma_{r-1}(v), \tag{11}$$

$$|\Gamma_r(u) \cap \Gamma_{r-1}(v)| = k_r / 2. \tag{12}$$

Condition (2) also forces equality in (7). If we write Eq. (7) in the form of (3) (from which it was derived) for the triple v, w and x, we get

$$\Lambda_{r-1}(v) \cup \Lambda_1(x) = \Lambda_r(w). \tag{13}$$

Now we define a relation R on V. For any $x, y \in V$, we define $x \mathrm{R} y$ if and only if

$$\Lambda_1(x) = \Lambda_1(y).$$

This is an equivalence relation. We show that each equivalince class spans a complete graph $K_{k_r/2}$, and that the quotient G/R is isomorphic to a cycle C_n with $n = 2d(G)$ or $n = 2d(G) + 1$.

Suppose $u \in V$, $v \in \Lambda_1(u)$, $v \bar{\mathrm{R}} u$ (where $\bar{\mathrm{R}}$ denotes the complement of R) and

$$x \in \Gamma_r(v) \cap \Gamma_{r+1}(u).$$

If $y \mathrm{R} x$, then

$$y \in \Gamma_r(v) \cap \Gamma_{r+1}(u).$$

Conversely, if

$$y \in \Gamma_r(v) \cap \Gamma_{r+1}(u),$$

then (11) and (13) imply that $y \mathrm{R} x$. Thus $\Gamma_r(v) \cap \Gamma_{r+1}(u)$ is a single equivalence class. Moreover any equivalence class is of this form for some u and v (given $x \in V$ we can choose a vertex $u \in \Gamma_{r+1}(x)$ and a path $(u, v, \ldots, x)$ of length $r + 1$ from u to x; then $u \bar{\mathrm{R}} v$ and the equivalence class containing x is precisely $\Gamma_r(v) \cap \Gamma_{r+1}(u)$).

Now we show that

$$| \Gamma_r(v) \cap \Gamma_{r+1}(u) | = k_r / 2.$$

This is equivalent to the following by (5) and (12).

$$\Gamma_r(v) \cap \Gamma_{r+1}(u) = \varnothing.$$

Let $(u, v, w, \ldots, z, x)$ be one of the shortest path from u to x. For any

$$a \in \Gamma_{r-1}(u) \cap \Gamma_r(v) \quad (\neq \varnothing \text{ by } (6)),$$

we have

$$\partial(w,a) = r+1 \qquad (14)$$

by (13) and $\partial(w,a) \leq r+1$. Hence there exists a path of length $r+1$ of the form

$$(w, v, u, \ldots, a)$$

from w to a. By (13) we have

$$\Lambda_{r-1}(v) \cup \Lambda_1(a) = \Lambda_r(u).$$

This implies

$$\Gamma_r(u) \cap \Gamma_r(v) \subset \Lambda_1(a).$$

Interchanging the role of u and v, we have $\partial(b,z)=1$ for any $b \in \Gamma_r(u) \cap \Gamma_r(v)$. Thus we get

$$\partial(w,a) \leq \partial(a,b) + \partial(b,z) + \partial(z,w) \leq r.$$

This contradicts (14). Hence we have

$$\Gamma_r(u) \cap \Gamma_r(v) = \emptyset,$$

so each equivalence class has size $k_r / 2$ as claimed. Finally, since

$$k_1 = 2(k_r / 2) + (k_r / 2 - 1)$$

the quotient graph has degree 2.

CHAPTER THREE

Intersection Diagrams of Distance-Regular Graphs

3.1. Preliminaries for Distance–Regular Graphs

Let $G = (V, E)$ be a connected graph. G is said to be *distance-regular* if the size of $D^r_s(u,v)$ depends only on the distance between u and v rather than the individual vertices. More precisely, G is distance-regular if the following equality holds for every integer r, s and for every vertices u, v, u', v' with $\partial(u,v) = \partial(u',v')$.

$$|D^r_s(u,v)| = |D^r_s(u',v')|.$$

In this chapter, we assume G is a distance-regular graph with the diameter $d = d(G)$. Since a distance-regular graph is also a distance degree regular, the r-th degree

$$k_r = |\Gamma_r(u)|$$

is independent on the choise of u. The 1-th degree $k = k_1$ of G is called the *valency* of G. For two vertices u, v with $\partial(u,v) = t$, we put

$$p^t_{rs} = |D^r_s(u,v)|.$$

The parameter p^r_{rs} is called the *intersection number* of G. Especially we put

$$a_r = p^r_{1\,r}, \quad b_r = p^r_{1\,r+1}, \quad c_r = p^r_{1\,r-1}.$$

These parameters a_r, b_r, c_r are also called the intersection numbers of G. Clearly

$$c_0 = 0, \quad a_0 = 0, \quad b_0 = k,$$
$$c_1 = 1, \quad b_d = 0,$$
$$a_r + b_r + c_r = k.$$

The *intersection array* of G is an array of the parameters a_r, b_r, c_r arranged as follows.

$$\left\{ \begin{matrix} 0 & 1 & c_2 & \cdots\cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots\cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots\cdots & b_{d-1} & 0 \end{matrix} \right\}$$

It is well known that if the intersection array of G is given then all intersection numbers $p^t_{r\,s}$ are determined uniquely. So the intersection numbers a_r, b_r, c_r are very important in the theory of distance-regular graphs. These parameters satisfy the following well-known condition.

$$0 = c_0 \leq c_1 \leq c_2 \leq \cdots\cdots \leq c_d,$$
$$k = b_0 \geq b_1 \geq b_2 \geq \cdots\cdots \geq b_d = 0.$$

The r-th degree k_r is given by the following formula.

$$k_r = |\Gamma_r(u)| = \frac{b_0 b_1 \cdots\cdots b_{r-1}}{c_1 c_2 \cdots\cdots c_r}.$$

We also have the following formula for an edge uv in G.

$$|D^r_r(u,v)| = \frac{k_r a_r}{k},$$

$$|D^{r+1}_r(u,v)| = |D^r_{r+1}(u,v)| = \frac{k_r b_r}{k} = \frac{k_{r+1} c_{r+1}}{k}.$$

The *girth* $g = g(G)$ is the length of a shortest cycle in G.

More precise descriptions about distance-regular graphs will be found in [1].

3.2. Intersection Diagrams of Distance-Regular Graphs

In this section we fix two vertices u, v in G and put

$$D_s^r = D_s^r(u,v), \quad e_\nu^\mu = e_\nu^\mu(x; u, v).$$

By lemma 1.3, we get the following lemma.

Lemma 3.1. *Let u, v be two vertices in G and let x be a vertex in D_s^r. Then we have the following relations.*

$$c_r = \sum_\nu e_\nu^{-1}(x), \quad a_r = \sum_\nu e_\nu^0(x), \quad b_r = \sum_\nu e_\nu^{+1}(x),$$

where ν ranges over $\{-1, 0, +1\}$. We get also

$$c_s = \sum_\mu e_{-1}^\mu(x), \quad a_s = \sum_\mu e_0^\mu(x), \quad b_s = \sum_\mu e_{+1}^\mu(x).$$

In the rest of this section, we assume $\partial(u,v) = 1$.

Lemma 3.2. *Let $x \in D_r^{r+1}$. Then the following equalities hold.*

(i) $e_{-1}^0(x) = e_{-1}^{+1}(x) = e_0^{+1}(x) = 0$,

(ii) $e_{+1}^{+1}(x) = b_{r+1}$, $\quad e_{+1}^{+1}(x) + e_{+1}^0(x) + e_{+1}^{-1}(x) = b_r$,

(iii) $e_{-1}^{-1}(x) = c_r$, $\quad e_{+1}^{-1}(x) + e_0^{-1}(x) + e_{-1}^{-1}(x) = c_{r+1}$,

(iv) $e_0^0(x) + e_{+1}^0(x) = a_{r+1}$, $\quad e_0^0(x) + e_0^{-1}(x) = a_r$.

Lemma 3.3.

(i) *If $b_r = b_{r+1}$ then there is no edge between D_r^{r+1} and $D_{r+1}^{r+1} \cup D_{r+1}^r$,*

(ii) *If $c_r = c_{r+1}$ then there is no edge between D_r^{r+1} and $D_{r+1}^r \cup D_r^r$.*

Lemma 3.4. *Let $x \in D_r^r$. Then the following equalities hold.*

(i) $e_{-1}^{+1}(x) = e_{+1}^{-1}(x) = 0$,

(ii) $e^{+1}_{+1}(x) + e^{+1}_{0}(x) = e^{+1}_{+1}(x) + e^{0}_{+1}(x) = b_r$,

(iii) $e^{-1}_{-1}(x) + e^{-1}_{0}(x) = e^{-1}_{-1}(x) + e^{0}_{-1}(x) = c_r$,

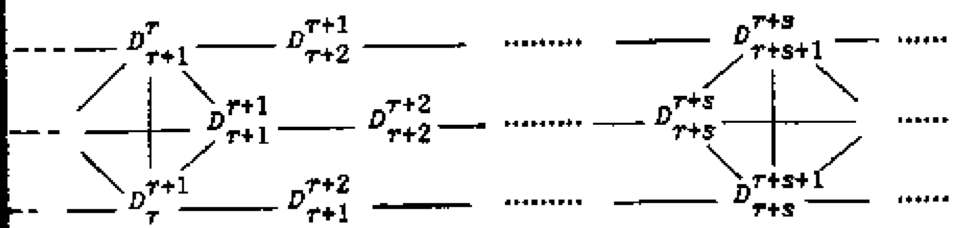
(iv) $e^{0}_{+1}(x) + e^{0}_{0}(x) + e^{0}_{-1}(x) = e^{+1}_{0}(x) + e^{0}_{0}(x) + e^{-1}_{0}(x) = a_r$.

Lemma 3.5. *Let r, s be positive integers. If*

$$b_{r+1} = b_{r+2} = \cdots\cdots = b_{r+s},$$

$$c_{r+1} = c_{r+2} = \cdots\cdots = c_{r+s}$$

hold, then the intersection diagram of G takes the following form.



CHAPTER FOUR

Some Applications of Intersection Diagrams

4.1. A Proof of Ivanov's Theorem

In [12], Ivanov proved a epoch-making result on distance-regular graphs. The main result is stated as follows.

Theorem 4.1 (Ivanov [12]). *Let G be a distance-regular graph with the intersection array*

$$\left\{ \begin{matrix} 0 & 1 & c_2 & \cdots\cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots\cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots\cdots & b_{d-1} & 0 \end{matrix} \right\}$$

Suppose

$$\begin{pmatrix} c_r \\ a_r \\ b_r \end{pmatrix} \neq \begin{pmatrix} c_{r+1} \\ a_{r+1} \\ b_{r+1} \end{pmatrix} = \begin{pmatrix} c_{r+2} \\ a_{r+2} \\ b_{r+2} \end{pmatrix} = \begin{pmatrix} c_{r+s} \\ a_{r+s} \\ b_{r+s} \end{pmatrix} .$$

Then we have $s \leq r+1$ *if* $r > 0$.

Corollary 4.2 (Ivanov [12]). *Let G be a distance-regular graph. Suppose the intersection array satisfies*

$$\begin{pmatrix} 1 \\ a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_2 \\ a_2 \\ b_2 \end{pmatrix} = \cdots\cdots = \begin{pmatrix} c_r \\ a_r \\ b_r \end{pmatrix} \neq \begin{pmatrix} c_{r+1} \\ a_{r+1} \\ b_{r+1} \end{pmatrix}.$$

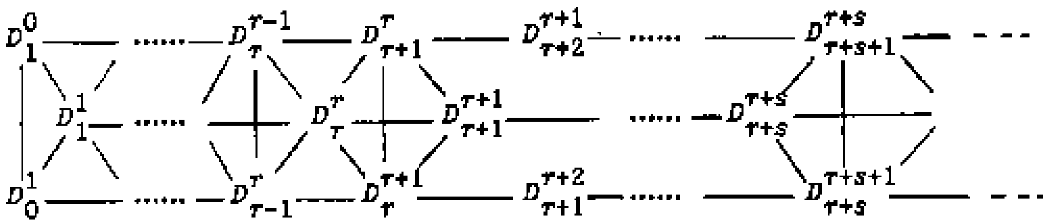
Then the diameter $d = d(G)$ is bounded by some function depending on r and the valency k. In particular, if the girth g is greater than 3, then the diameter d is bounded by a certain function depending on k and g.

Ivanov's original proof is fairly difficult to read though it is short. We noticed that Theorem 4.1 is easily proved by using the intersection diagram. Here, we prove Theorem 4.1 by useing the intersection diagram.

Proof of Theorem 4.1. Fix an edge uv in G, and we consider the intersection diagram of G with respect to (u,v). Put

$$D_s^r = D_s^r(u,v).$$

By way of contradiction, we assume $s \geq r + 2$. Then the intersection diagram takes the following form by Lemma 3.5.



Take a vertex x in D_r^{r+1}. Since $(c_r, a_r, b_r) \neq (c_{r+1}, a_{r+1}, b_{r+1})$, there must be a vertex y in $D_r^r \cup D_{r+1}^r \cup D_{r+1}^{r+1}$ which is adjacent to x, by Lemma 3.2.

First we suppose $y \in D_{r+1}^{r+1}$. Chose $z \in D_{2r+1}^{2r+1}$ such that $\partial(y,z) = r$;

this is possible because there are b_{r+1} edges from y to D^{r+2}_{r+2} and so on. Since $s \geq r+2$, the shape of the intersection diagram implies that the b_{r+1} vertices in D^{r+2}_{r+1} adjacent to x have distance $r+2$ from z. Since $\partial(x,z) = r+1$, there cannot be any more vertices which are adjacent to x have distance $r+2$ apart from z, which is a contradiction since there is at least one edge from x to D^{r}_{r-1}.

Next we suppose $y \in D^{r}_{r+1}$. Choose $z \in D^{2r}_{2r+1}$ such that $\partial(y,z) = r$. Then we get a contradiction by same argument as above. So there is no edge from x to $D^{r+1}_{r+1} \cup D^{r}_{r+1}$.

Last we suppose $y \in D^{r}_{r}$. Choose z either in D^{2r-1}_{2r} or in D^{2r}_{2r} such that $\partial(y,z) = r$. This implies a contradiction as above.

2.2. An Inequality Between Intersection Numbers

In this section we prove the following inequalities between intersection numbers of a distance-regular graph.

Theorem 4.3. *Let G be a distance-regular graph with diameter d and intersection numbers a_r, b_r, c_r. Then for every integer r with $0 < r < d$, the following inequalities hold.*

(i) $a_{r+1} \geq a_r(1 - (a_r / b_r))$,

(ii) $a_r \geq a_{r+1}(1 - (a_{r+1} / c_{r+1}))$.

Corollary 4.4. *For every integer i with $0 < r < d$, the following hold.*

(i) *If* $0 < a_r < b_r$ *then* $a_{r+1} > 0$.

(ii) *If* $0 < a_{r+1} < c_{r+1}$ *then* $a_r > 0$.

Corollary 4.4 is a direct consequence of Theorem 4.3. In the following proof, $e(X, Y)$ denotes the number of edges between two subsets X and Y of the vertex set V.

Proof of Theorem 4.3. We fix an edge uv in G and we consider the intersection diagram of G with respect to (u, v). Put

$$D_s^r = D_s^r(u,v).$$

Now we count the number of edges between D_r^r and D_r^{r+1}. For $x \in D_r^r$, the number of edges connecting x and D_r^{r+1} is at most a_r. On the other hand, for $y \in D_r^{r+1}$, we have

$$e\,(y\,,D_r^r \cup D_r^{r+1}) \;=\; a_r\,,$$
$$e\,(y\,,D_{r+1}^{r+1} \cup D_r^{r+1}) \;=\; a_{r+1}\,.$$

So we get

$$e\,(y\,,D_r^r) \;=\; a_r - a_{r+1} + e\,(y\,,D_{r+1}^{r+1}) \;\geq\; a_r - a_{r+1}\,.$$

Thus

$$a_r\,|D_r^r| \;\geq\; e\,(D_r^r\,,D_r^{r+1}) \;\geq\; (a_r - a_{r+1})\,|D_r^{r+1}|\,.$$

Here we use

$$|D_r^r| = k_r a_r / k \quad \text{and} \quad |D_r^{r+1}| = k_r b_r / k.$$

Then we get

$$a_r^2 \geq (a_r - a_{r+1})\,b_r,$$

and this implies (*i*) of Theorem 4.3.

Similarly, (ii) of Theorem 4.3 may be proved by counting the number of edges between D_{r+1}^{r+1} and D_r^{r+1}.

4.3. A Theorem on Intersection Arrays

Let $G = (V, E)$ be a distance-regular graph with vertex set V and edge set E. Let a_r, b_r, c_r be the intersection numbers of G. By way of recourse to the inequalities $c_i \leq c_{i+1}$ and $b_i \geq b_{i+1}$, we may write the intersection array of G in the following form.

$$\left\{ \begin{array}{cccccccccc} * & 1 & \cdots\cdots & 1 & 1 & \cdots\cdots & 1 & \cdots\cdots \\ 0 & 0 & \cdots\cdots & 0 & 1 & \cdots\cdots & 1 & \cdots\cdots \\ k & k-1 & \cdots\cdots & k-1 & k-2 & \cdots\cdots & k-2 & \cdots\cdots \end{array} \right\}$$

Remark that the number of columns of type $(1,0,k-1)$ is at least one if the girth of G is greater than three. We obtained the following theorem.

Theorem 4.5. *Let G be a distance-regular graph with the girth greater than three. Then the number of columns of type* $(1, 1, k-2)$ *in the intersection array of G is at most four.*

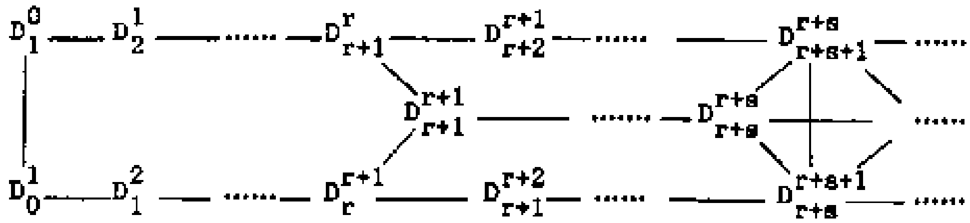
Recently, Biggs, Boshier and Shawe-Taylor completed the classification of distance-regular graphs of valency three ([4]). The key of their proof is to show that the number of columns of type $(1,1,1)$ is at most three in any distance-regular graph with valency three and girth greater than three. Theorem 4.5 is a partial extension of this fact.

Let $G = (V,E)$ be a distance-regular graph with valency $k \geq 3$. Number of columns of type $(1,0,k-1)$, $(1,1,k-2)$ in the intersection array will be denoted by r and s respectively. We assume G has girth greater than three, so we have $r > 0$.

We fix an edge uv in G and we consider the intersection diagram of G with respect to (u,v). We put

$$D^{\tau}_{s} = D^{\tau}_{s}(u,v),$$
$$e^{\mu}_{\nu}(x) = e^{\mu}_{\nu}(x;\, u,\, v).$$

Then the intersection diagram of G takes the form



by lemmas in chapter 3.

By way of contradiction, we assume $s > 4$. Note that $r+1 \geq s$ by Theorem 4.1.

First we determine the edge patterns $e^{\mu}_{\nu}(x)$ for various x. For a vertex x and a subset Y in V, let $e(x, Y)$ denote the number of edges which connect x and vertices in Y.

Proposition 4.6.

(i) *If* $x \in D^{i+1}_{i}$ *for* $0 < i < r$, *then*

$$e(x, D^{i}_{i-1}) = 1 \quad and \quad e(x, D^{i+2}_{i+1}) = k-1.$$

(ii) *If* $x \in D^{r+1}_{r}$ *then*

$$e(x, D^{r}_{r-1}) = e(x, D^{r+1}_{r+1}) = 1 \quad and \quad e(x, D^{r+2}_{r+1}) = k-2.$$

(iii) *If* $x \in D^{i+1}_{i}$ *for* $r < i < r+s$ *then*

$$e(x, D^{i}_{i-1}) = e(x, D^{i+1}_{i}) = 1 \quad and \quad e(x, D^{i+2}_{i+1}) = k-2.$$

(iv) *If* $x \in D_{r+1}^{r+1}$ *then*

$$e(x, D_r^{r+1}) = e(x, D_{r+1}^r) = 1 \quad and \quad e(x, D_{r+2}^{r+2}) = k-2.$$

(v) *If* $x \in D_i^i$ *for* $r+1 < i < r+s$ *then*

$$e(x, D_{i-1}^{i-1}) = e(x, D_i^i) = 1 \quad and \quad e(x, D_{i+1}^{i+1}) = k-2.$$

Proof. We shall only prove (iv), the other cases follow along similar lines. Let $x \in D_{r+1}^{r+1}$. By Lemma 3.4,

$$e(x, D_{r+2}^{r+1}) + e(x, D_{r+2}^{r+2}) = b_{r+1} = k-2.$$

Since there is no edge between D_{r+1}^{r+1} and D_{r+2}^{r+1}, we have

$$e(x, D_{r+2}^{r+1}) = 0.$$

Therefore

$$e(x, D_{r+2}^{r+2}) = k-2.$$

Again by Lemma 3.4,

$$e(x, D_{r+1}^{r}) + e(x, D_r^r) = c_r = 1.$$

But now D_r^r is empty. Thus,

$$e(x, D_{r+1}^r) = 1.$$

Similarly we get also

$$e(x, D_r^{r+1}) = 1.$$

For a cycle

$$C : x_0 , x_1 , \dots , x_{m-1}$$

in G, we consider the profile of C which has been defined in [3]. We give a slightly different definition. Let $\{D_j^i\}$ be the intersection diagram of G with respect to the edge (x_0, x_1). Then each x_t $(0 \leq t < m)$ is contained in some D_j^i. Put $D(t) = D_j^i$. Then we get a series

$$D(0), D(1), \dots\dots , D(m-1)$$

which will be called the *profile* of the cycle C with respect to (x_0, x_1).

For example, take an edge (x_0, x_1) of G and consider the intersection diagram $\{D_j^i\}$ with respect to (x_0, x_1). Take an edge (x_{r+2}, x_{r+3}) in D_{r+1}^{r+2}, take x_{r+4} in $\Gamma_1(x_{r+3}) \cap D_r^{r+1}$ and take x_{r+5} in $\Gamma_1(x_{r+4}) \cap D_{r+1}^{r+1}$. Connect x_1 and x_{r+2} by a $(r+1)$-path

$$x_1\ ,\ x_2\ ,\ \dots\dots\ ,\ x_{r+2}$$

and connect x_{r+5} and x_0 by a $(r+1)$-path

$$x_{r+5}\ ,\ \dots\dots\ ,\ x_{2r+5}\ ,\ x_0\ .$$

Then we get a $(2r+6)$-cycle

$$x_0\ ,\ x_1\ ,\ \dots\dots\ ,\ x_{2r+5}$$

and the pofile of C with respect to (x_0, x_1) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+2}, D_{r+1}^{r+2}, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1 .$$

Now we determine the profiles of C in the above example with respect to (x_0, x_1), (x_1, x_2), (x_2, x_3), ... By the form of the intersection diagram and by the proposition, the profile of C with respect to (x_1, x_2) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, D_{r+2}^{r+1}, D_{r+2}^{r+1}, D_{r+1}^{r}, \dots, D_2^1$$

where $\{D_j^i\}$ denotes the intersection diagram with respect to (x_1, x_2). The profile of C with respect to (x_2, x_3) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+2}^{r+2}, D_{r+2}^{r+2}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1 ,$$

and the profile with respect to (x_3, x_4) is

$$D_1^0, D_0^1, D_r^{r+1}, D_{r+1}^{r+2}, D_{r+1}^{r+2}, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1 .$$

But the profile with respect to (x_3, x_4) is same as the profile with respect to (x_0, x_1). This means the length of the cycle must be a multiple of 3. Hence we have $2r+6 \equiv 0 \pmod 3$, $r \equiv 0 \pmod 3$.

To get another condition on r, we take a $(2r+13)$-cycle

$$C' : y_0 , y_1 , \ldots\ldots , y_{2r+12}$$

whose profile with respect to (y_0, y_1) is

$$D_1^0 , D_0^1 , \ldots , D_r^{r+1} , D_{r+1}^{r+2} , D_{r+2}^{r+3} , D_{r+2}^{r+3} , D_{r+1}^{r+2} , D_r^{r+1} ,$$
$$D_{r+1}^{r+1} , D_{r+2}^{r+2} , D_{r+3}^{r+3} , D_{r+3}^{r+3} , D_{r+2}^{r+2} , D_{r+1}^{r+1} , D_{r+1}^{r} , \ldots , D_2^1 .$$

We calculate the profile of C' with respect to (y_1, y_2), (y_2, y_3), ... *The profiles with respect to (y_1, y_2) and (y_2, y_3) are determined uniquely. But the profile with respect to (y_3, y_4) has two possibilities,* and we must calculate the profiles with respect to (y_3, y_4), (y_4, y_5), ... in each case separately. Fortunately, the profiles with respect to (y_7, y_8) coincide in each case. Thus, the profiles with respect to (y_7, y_8) and (y_8, y_9) are uniquely determined. Again the profiles with respect (y_9, y_{10}), ... , (y_{12}, y_{13}) have two possibilities. But the profiles with respect to (y_{13}, y_{14}) are coincident, and the profiles with respect to (y_{13}, y_{14}), (y_{14}, y_{15}) and (y_{15}, y_{16}) are uniquely determined. The profile with respect to (y_{15}, y_{16}) coincides with the profile with respect to (y_0, y_1). Therefore $2r+13 \equiv 0 \pmod{15}$, $r \equiv 1 \pmod{3}$. This is a contradiction.

Remark. Two cycles in the above proof are the same as those used in [4]. But the profiles of the $(2r+13)$-cycle is not uniquely determined in our case, i.e. that does not have a *good profile* in terms of [4].

CHAPTER FIVE

Distance-Regular Graphs with Valency Four and Girth Three

5.1. The Classification Theorem

In this chapter we shall classify distance-regular graphs with valency four and girth three.

Theorem 5.1. *Let G be a distance-regular graph with valency four and girth three. Then G is isomorphic to one of the following graphs.*

(i) *complete graph* K_5

(ii) *Octahedron*

(iii) *The line graph of one of the following graphs with valency* 3.

(a) *Petersen's graph* O_3 (b) *complete bipartite graph* $K_{3,3}$

(c) *Heawood graph* (d) 8-*cage* (e) 12-*cage*

Recently, N.L. Biggs, A. Boshier and J. Shawe-Taylor completed the classification of distance-regular graphs with valency 3 ([4]). Discussions about significance of classifying distance-regular graphs will be found in the book by Bannai and Ito ([1]). In the proof of the above theorem we shall only use pure combinatorial method, though there is an exception that we shall use the results by Bannai and Ito ([2]), Damerell ([7]), Feit and Higman ([8]), whose proofs require algebraic methods.

5.2. Locally Triangular Graphs

Let G be a connected simple graph G, not neccessarily distance-regular, with the vertex set V. G is said to be *locally triangular* if for any edge (x,y) in G there exists just one vertex z which is adjacent to both x and y. The *triangle graph* $\overline{G}$ of a locally triangular graph G is a graph with the vertex set

$\overline{V} = \{ \{u_1, u_2, u_3\} \mid u_1, u_2, u_3 \in V, u_1, u_2, u_3$ are adjacent to each other $\}$,

and two vertices $\overline{u} = \{u_1, u_2, u_3\}$, $\overline{v} = \{v_1, v_2, v_3\}$ are defined to be adjacent in $\overline{G}$ if $\overline{u} \neq \overline{v}$ and $\overline{u} \cap \overline{v} \neq \emptyset$ hold.

There is an usual metric $\overline{\partial}$ on $\overline{G}$ which is defined as the length of a shortest path between two vertices of $\overline{G}$. There is another metric ∂ on $\overline{G}$ defined by

$$\partial(\overline{u}, \overline{v}) = min \{ \partial(u, v) \mid u \in \overline{u}, v \in \overline{v} \}.$$

A relation between ∂ and $\overline{\partial}$ is given by the following lemma.

Lemma 5.2. *Let G be a locally triangular graph and let $\overline{u}$, $\overline{v}$ be two distinct vertices in $\overline{G}$. Then*

$$\overline{\partial}(\overline{u}, \overline{v}) = \partial(\overline{u}, \overline{v}) + 1.$$

Proof. First we assume $\overline{\partial}(\overline{u}, \overline{v}) = r$, $\overline{u}, \overline{v} \in \overline{G}$. Then there is a path in $\overline{G}$ connecting $\overline{u}$ and $\overline{v}$: $\overline{u} = \overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots\ldots, \overline{x}_r = \overline{v}$. Take $x_i \in \overline{x}_i \cap \overline{x}_{i+1}$ $(0 \leq i \leq r-1)$. Then x_i and x_{i+1} are adjacent in G, since x_i and x_{i+1} are both in $\overline{x}_{i+1}$. This implies $\partial(x_0, x_{r-1}) \leq r-1$, $\partial(\overline{u}, \overline{v}) \leq r-1$. Hence we get $\overline{\partial}(\overline{u}, \overline{v}) \geq \partial(\overline{u}, \overline{v}) + 1$. Next we assume $\partial(\overline{u}, \overline{v}) = r$. Take $u \in \overline{u}$ and $v \in \overline{v}$ with $\partial(u, v) = r$. Let $u = x_0, x_1, x_2, \ldots\ldots, x_r = v$ be a path of length r connecting u and v. Take a vertex z_i which is adjacent to both x_i and x_{i+1} $(0 \leq i \leq r-1)$, and put $\bar{x}_i = \{x_i, x_{i+1}, z_i\}$. Then we get a series of vertices in $\bar{G}$: $\bar{x}_0$, $\bar{x}_1$, $\bar{x}_2$, ... , $\bar{x}_{r-1}$. Since x_{i+1} belongs to both $\bar{x}_i$ and $\bar{x}_{i+1}$, we have $\bar{x}_i \cap \bar{x}_{i+1} \neq \emptyset$. This implies $\bar{\partial}(\bar{x}_0, \bar{x}_{r-1}) \leq r-1$. Hence we have $\bar{\partial}(\bar{u}, \bar{v}) \leq r+1$, since $\bar{u}$ is adjacent to $\bar{x}_0$ and $\bar{v}$ is adjacent to $\bar{x}_{r-1}$ in $\bar{G}$. So we get $\bar{\partial}(\bar{u}, \bar{v}) \leq \partial(\bar{u}, \bar{v}) + 1$.

Lemma 5.3. *Let G be a locally triangular graph. If G is regular of degree four, then $L(\bar{G})$ is isomorphic to G.*

Proof. First we remark that for any vertex x in G there is just two vertices of $\bar{G}$ which include x, since degree of x is four and G is locally triangular.

Let $(\bar{u}, \bar{v})$ be an edge in $\bar{G}$, then $|\bar{u} \cap \bar{v}| = 1$, since G is locally triangular. Let f be a mapping of $L(\bar{G})$ to G which is defined as

$$f(\bar{u}, \bar{v}) = x \ , \quad x \in \bar{u} \cap \bar{v}.$$

Take any vertex x in G. There are two vertices $\bar{u}$, $\bar{v}$ in $\bar{G}$ with $x \in \bar{u}$ and $x \in \bar{v}$. Then we have $f(\bar{u}, \bar{v}) = x$. So f is onto. Since f is clearly one-to-one by the above remark, f is a bijection.

To see that f is an isomorphism between $L(\bar{G})$ and G, take two adjecent vertices $(\bar{u}, \bar{v})$, $(\bar{u'}, \bar{v'})$ in $L(\bar{G})$. By definition of a line graph, we may assume $\bar{u} = \bar{u'}$. Let $f(\bar{u}, \bar{v}) = x$ and $f(\bar{u}, \bar{v'}) = x'$. Since $x, x' \in \bar{u}$, x is adjacent to x' in G. So f maps adjacent vertices in $L(\bar{G})$ to adjacent vertices in G. It is easy to show that if $f(\bar{u}, \bar{v})$ is adjacent to $f(\bar{u'}, \bar{v'})$ then $(\bar{u}, \bar{v})$ is adjacent to $(\bar{u'}, \bar{v'})$. Hence f is an isomorphism.

5.3. Intersection Arrays of Locally Triangular Graphs

Let G be a locally triangular distance-regular graph with valency four. Let d be the diameter of G and a_r, b_r, c_r $(0 \leq r \leq d)$ be the intersection numbers of G. Remark that $a_1 = 1$ since G is locally triangular.

We may assume that the intersection array of G takes the following form, by $b_r \geq b_{r+1}$, $c_r \leq c_{r+1}$ and $a_r + b_r + c_r = 4$.

$$\left\{\begin{array}{cccccccccccccccccccc} 0 & 1 & \dots & 1 & 1 & \dots & 1 & 2 & \dots & 2 & 2 & \dots & 2 & 3 & \dots & 3 & c_d \\ 0 & 1 & \dots & 1 & 2 & \dots & 2 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 & a_d \\ 4 & 2 & \dots & 2 & 1 & \dots & 1 & 2 & \dots & 2 & 1 & \dots & 1 & 1 & \dots & 1 & 0 \end{array}\right\}.$$

Since $a_1 = c_1 = 1$, the number of colums of type (1,1,2) is at least one. The colums of type (1,2,1) and the columns of type (2,0,2) do not appear at the same time.

Lemma 5.4. *There is no column of type (2,0,2) in the intersection array of G.*

Proof. Let a be the number of columns of type (1,1,2). Assume there is a column of type (2,0,2). Let u be a vertex in G and take a vertex x in $\Gamma_{\alpha+1}(u)$. There is an edge (x, y) with $y \in \Gamma_\alpha(u)$. Since G is locally triangular, there is a vertex z which is adjacent to x and y. Then we get $z \in \Gamma_\alpha(u)$ by $a_{\alpha+1} = 0$. Moreover we take an edge (y,w) with $w \in \Gamma_{\alpha-1}(u)$. Since G is locally triangular, there is a vertex v which is adjacent to w and y. Then we get $v \in \Gamma_\alpha(u)$ by $c_\alpha = 1$. Then we have two edges (y,z) and (y,v). But since $a_\alpha = 1$, we get $z = v$. Then there is two vertices w, x which are adjacent to y and z, a contradiction.

Lemma 5.5. *There is no column of type* (3,0,1).

Proof. This follows from the fact that 3 does not divide the size of $\Gamma_r(u)$.

LEMMA 5.6. *There is at most one column of type* (1,2,1).

Proof. Assume there is at least two columns of type (1,2,1). Let α be the number of columns of type (1,1,2). Let u be a vertex in G and x be a vertex in $\Gamma_{\alpha+1}(u)$. Take an edge (x,y) with $y \in \Gamma_{\alpha+2}(u)$. Since G is locally triangular, there is a vertex z which is adjacent to x and y. Since $b_{\alpha+1} = 1$, z is not in $\Gamma_{\alpha+2}(u)$, so we have $z \in \Gamma_{\alpha+1}(u)$. This contradicts to $c_{\alpha+2} = 1$.

Lemma 5.7. *If there is no column of type* (1,2,1) *then there is no column of type* (2,1,1).

Proof. Assume there is no column of type (1,2,1), but there is a column of type (2,1,1). Let α be the number of columns of type (1,1,2) and let u be a vertex in G. By Lemma 5.4, we have $c_{\alpha+1} = 2$, $a_{\alpha+1} = 1$ and $b_{\alpha+1} = 1$. Take a vertex x in $\Gamma_{\alpha+1}(u)$. Let $\Gamma_1(x) = \{y_1, y_2, z, w\}$ with $y_1, y_2 \in \Gamma_\alpha(u)$, $z \in \Gamma_{\alpha+1}(u)$, $w \in \Gamma_{\alpha+2}(u)$. Since G is locally triangular, z is adjacent to w, y_1 is adjacent to y_2. Take an edge (y_1,p) with $p \in \Gamma_{\alpha-1}(u)$. Since G is locally triangular, there is a vertex q which is adjacent to p and y_1. We have $q \in \Gamma_\alpha(u)$ by $c_\alpha = 1$. But since $a_\alpha = 1$, we get $q = y_2$. Then there is two vertices p, x which is adjacent to y_1, y_2. This is a contradiction.

Lemma 5.8. *Let (u,v) be an edge in G and put $e_\nu^\mu(x) = e_\nu^\mu(x; u, v)$. Let α be the number of columns of type (1,1,2). Then we have the followings.*

(i) *For $x \in D_1^1$, we have*

$$e_{-1}^{0}(x) = e_0^{-1}(x) = 1 \text{ and } e_{+1}^{+1}(x) = 2.$$

(ii) *For $x \in D_r^{r+1}$ or D_{r+1}^{r} $(1 \leq r \leq \alpha-1)$, we have*

$$e_{-1}^{-1}(x) = e_0^{0}(x) = 1 \text{ and } e_{+1}^{+1}(x) = 2.$$

(iii) *For $x \in D_r^r$ $(2 \leq r \leq \alpha)$, we have*

$$e_{-1}^{-1}(x) = e_0^{0}(x) = 1 \text{ and } e_{+1}^{+1}(x) = 2.$$

Assume there is a column of type (1,2,1). Then

(iv) *For $x \in D_\alpha^{\alpha+1}$ we have*

$$e_{-1}^{-1}(x) = e_0^{0}(x) = e_{+1}^{0}(x) = e_{+1}^{+1}(x) = 1.$$

Assume the columns of type (1,2,1) and (2,1,1) both exist. Then

(v) *For $x \in D_{\alpha+1}^{\alpha+2}$*

$$e_{-1}^{-1}(x) = e_0^{-1}(x) = e_0^{0}(x) = e_{+1}^{+1}(x) = 1.$$

(vi) *For $x \in D_{\alpha+1}^{\alpha+1}$, one of the followings holds.*

(a) $e_0^{-1}(x) = e_{-1}^{0}(x) = e_{+1}^{0}(x) = e_0^{+1}(x) = 1.$

(b) $e_{-1}^{-1}(x) = e_{+1}^{+1}(x) = 1$ and $e_0^{0}(x) = 2.$

Proof. These are the consequences of the lemmas described in section 2. In the proof of (iv) (when $r = \alpha$) and (vi), we need the fact that G is locally triangular.

Lemma 5.9. *There is no column of type* (2,1,1).

Proof. Assume there is at least one column of type (2,1,1). Then there is at least one column of type (1,2,1) by Lemma 5.7. Let α be the number of columns of type (1,1,2).

We claim any $(\alpha+1)$-path $z_0, z_1, z_2, \dots\dots, z_{\alpha+1}$ with $\partial(z_0, z_{\alpha+1}) = \alpha+1$ can be extended to a $(2\alpha+3)$-cycle. Put $D_s^\tau = D_s^\tau(z_0, z_1)$, $e_\nu^\mu(x) = e_\nu^\mu(x; z_0, z_1)$. Then $z_\tau \in D_{\tau-1}^\tau$ for $(1 \leq \tau \leq \alpha+1)$. By Lemma 5.8 we have $e_{+1}^0(z_{\alpha+1}) = 1$, hence there is a vertex $z_{\alpha+2}$ in $D_{\alpha+1}^{\alpha+1}$ which is adjacent to $z_{\alpha+1}$. Since $e_0^{-1}(z_{\alpha+2}) = 1$ by Lemma 5.8, there is a vertex $z_{\alpha+3}$ in $D_{\alpha+1}^{\alpha}$ which is adjacent to $z_{\alpha+2}$. Since $\partial(z_{\alpha+3}, z_0) = \alpha$, we can take a path $z_{\alpha+3}$, $z_{\alpha+4}, \dots\dots, z_{2\alpha+3} = z_0$. Clearly $z_{\alpha+3+i} \in D_{\alpha-i+1}^{\alpha-i}$ for $0 \leq i \leq \alpha$. So we get a $(2\alpha+3)$-cycle $z_0, z_1, \dots\dots, z_{2\alpha+3} = z_0$, as required.

Now we fix an edge (u,v) and we put $D_s^\tau = D_s^\tau(u,v)$, $e_\nu^\mu(x) = e_\nu^\mu(x; u, v)$. Put $z_0 = v$ and take a vertex z_1 in D_1^1. Since $e_{+1}^{+1}(z_1) = 2$ by Lemma 5.8, there is a vertex z_2 in D_2^2 which is adjacet to z_1. For $2 \leq \tau \leq \alpha$, we have $e_{+1}^{+1}(z_\tau) = 1$, so we can take a vertex $z_{\tau+1}$ in $D_{\tau+1}^{\tau+1}$ which is adjacent to z_τ. We can extend the $(\alpha+1)$-path $z_0, z_1, \dots\dots, z_{\alpha+1}$ to a $(2\alpha+3)$-cycle $z_0, z_1, \dots\dots, z_{2\alpha+3}$ as we claimed above. Since $e_{-1}^{-1}(z_{\alpha+1}) = e_{+1}^{+1}(z_{\alpha+1}) = 1$ and $e_0^0(z_{\alpha+1}) = 2$, $z_{\alpha+2}$ must be in $D_{\alpha+1}^{\alpha+1}$ or $D_{\alpha+2}^{\alpha+2}$. But since $\partial(z_{\alpha+2}, z_0) \leq \alpha+1$, we have $z_{\alpha+2} \in D_{\alpha+1}^{\alpha+1}$. Since $e_{-1}^{-1}(z_{\alpha+2}) = e_{+1}^{+1}(z_{\alpha+2}) = 1$, $e_0^0(z_{\alpha+2}) = 2$ and $\partial(z_{\alpha+3}, z_0) \leq \alpha$, we get that $z_{\alpha+3}$ is in D_α^α. Similary we have $z_{\alpha+3+i} \in D_{\alpha-i}^{\alpha-i}$ for $(0 \leq i \leq \alpha-1)$. Especially we get $z_{2\alpha+2} \in D_1^1$. This is a contradiction since $|D_1^1| = a_1 = 1$.

Lemma 5.10. *If there is a column of type* (1,2,1), *then* $c_d = 4$.

Proof. Let α be the number of columns of type (1,1,2). Remark that $d = \alpha+2$ by the above lemmas. Take an edge (u,v) and put $D_s^{\tau} = D_s^{\tau}(u,v)$, $e_{\nu}^{\mu}(x) = e_{\nu}^{\mu}(x; u, v)$. Since $D_{\alpha}^{\alpha} \neq \varnothing$ and $e_{+1}^{+1}(y) = 2$ for $y \in D_{\alpha}^{\alpha}$, there is an edge (y,x) with $x \in D_{\alpha+1}^{\alpha+1}$. If $e_0^{+1}(x) = 0$, then we get a condradiction as in the proof of Lemma 5.9. So we have $e_0^{+1}(x) = e_{+1}^{0}(x) = 1$, $e_0^0(x) = 1$. Since G is locally triangular, there is a vertex z which is adjacent to x and y. We have $z \in D_{\alpha+1}^{\alpha+1}$ by $e_{-1}^{-1}(x) = 1$ and $e_{-1}^{0}(x) = e_0^{-1}(x) = 0$. Take an edge (x,p) with $p \in D_{\alpha+2}^{\alpha+1}$, and take a vertex q which is adjacent to x and p. Then $q \in D_{\alpha+1}^{\alpha+2}$. There is an edge (p,f) with $f \in D_{\alpha+1}^{\alpha}$ since $e_{-1}^{-1}(p) = c_{\alpha+1} = 1$. Take a vertex h which is adjacent to p and f. Then h is not in $D_{\alpha+2}^{\alpha+1}$ since $e_{+1}^{+1}(f) = 1$. So we have $h \in D_{\alpha+1}^{\alpha}$ or $h \in D_{\alpha+1}^{\alpha+1}$, in any way $h \in \Gamma_{\alpha+1}(v)$. Hence we have four edges (p,f), (p,x), (p,h), (p,q) with f, x, h, $q \in \Gamma_{\alpha+1}(v)$ and $p \in \Gamma_{\alpha+2}(v)$. This implies $c_{\alpha+2} = 4$.

Lemma 5.11. *If there is no column of type* (1,2,1) *then* $c_d = 2$.

Proof. Let α be the number of columns of type (1,1,2), then $d = \alpha + 1$. Let u be a vertex in G.

We have $c_d \neq 3$ since $|\Gamma_{\alpha}(u)|$ does not divided by 3. Assume $c_d = 4$. Take a vertex x in $\Gamma_{\alpha}(u)$ and take y and z in $\Gamma_{\alpha+1}(u)$ which are adjacent to x. Then there is a vertex w which is adjacent to x and y, and there is a vertex p which is adjacent to x and z. We have $w, p \in \Gamma_{\alpha}(u)$ since $a_d = 0$. Since $a_{\alpha} = 1$, we have $w = p$. Then y and z are adjacent to x and w, a contradiction. So we have $c_d \neq 4$.

Now we assume $c_d = 1$. Fix an edge (u,v) and put $e_{\nu}^{\mu}(x) = e_{\nu}^{\mu}(x; u, v)$, $D_s^{\tau} = D_s^{\tau}(u,v)$. Let x be a vertex in $D_{\alpha+1}^{\alpha+1}$. We have the following two possibilities.

(i) $e_0^{-1}(x) = e_{-1}^0(x) = 1$, $e_0^0(x) = 2$.

(ii) $e_{-1}^{-1}(x) = 1$, $e_0^0(x) = 3$.

Put $A = \{ x \in D_{\alpha+1}^{\alpha+1} \mid e_0^0(x) = 2 \}$, $B = \{ x \in D_{\alpha+1}^{\alpha+1} \mid e_0^0(x) = 3 \}$.

Assume there is an edge (x,y) with $x \in A$, $y \in B$. Since $e_{-1}^{-1}(y) = 1$, there is an edge (y,z) with $z \in D_\alpha^\alpha$. Take edges (x,p), (x,q) with $p \in D_{\alpha+1}^\alpha$, $q \in D_\alpha^{\alpha+1}$. Since G is locally triangular, there is a vertex w_1 which is adjacent to x and p. Also we can take a vertex w_2 which is adjacent to x and q. Clearly $w_1 \neq w_2$. Hence $y = w_1$ or $y = w_2$. We may assume $y = w_1$. Then there is two edges (y,p), (y,z) with $y \in \Gamma_{\alpha+1}(u)$ and $p, z \in \Gamma_\alpha(u)$, this contradicts to $c_\alpha = 1$. Hence there is no edge between A and B.

By Lemma 5.3, there is no edge between $X = \bigcup_{1 \leq \tau \leq \alpha} D_\tau^\tau$ and $Y = \bigcup_{1 \leq \tau \leq \alpha} D_\tau^{\tau+1}$. Hence there is no edge between $X \cup B$ and $Y \cup A$. This implies that there is no path of length $\leq d$ between B and D_1^2, a contradiction.

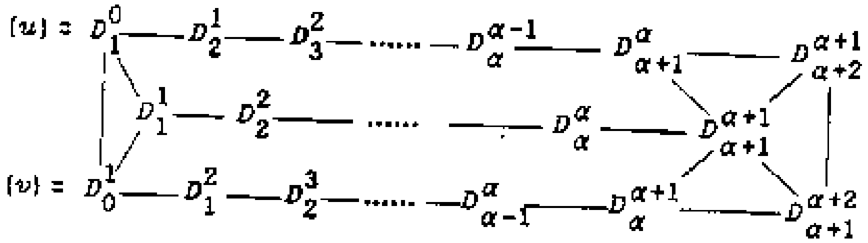
5.4. Proof of The Classification Theorem

In this section, we shall use the notations in section 5.3. Assume $a_1 = 1$. Then by the lemmas proved in section 4, the intersection array of G takes one of the following forms.

$$\textit{type } 1: \left\{\begin{matrix} 0 & 1 & \ldots\ldots & 1 & 1 & 4 \\ 0 & 1 & \ldots\ldots & 1 & 2 & 0 \\ 4 & 2 & \ldots\ldots & 2 & 1 & 0 \end{matrix}\right\}, \quad \textit{type } 2: \left\{\begin{matrix} 0 & 1 & \ldots\ldots & 1 & 2 \\ 0 & 1 & \ldots\ldots & 1 & 2 \\ 4 & 2 & \ldots\ldots & 2 & 0 \end{matrix}\right\}$$

Proposition 5.12. *If the intersection array of G is of type 1, the triangle graph $\overline{G}$ of G is a Moore graph with valency three.*

Proof. Let α be the number of columns of type (1,1,2) in the intersection array of G. Fix an edge (u,v) and put $D_s^r = D_s^r(u,v)$, $e_\nu^\mu(x) = e_\nu^\mu(x;\, u,\, v)$. By the lemmas described in Chapter 3, the intersection diagram of G takes the following form.



The numbers $e_\nu^\mu(x)$ are determined as follows.

(*i*) For $x \in D_1^1$, $e_0^{-1}(x) = e_{-1}^0(x) = 1$, $e_{+1}^{+1}(x) = 2$.

(*ii*) For $x \in D_{r+1}^r$, D_r^{r+1}, $(1 \leq r \leq \alpha-1)$ $e_{-1}^{-1}(x) = e_0^0(x) = 1$, $e_{+1}^{+1}(x) = 2$.

(*iii*) For $x \in D_{\alpha+1}^\alpha$, $e_{-1}^{-1}(x) = e_0^0(x) = e_0^{+1}(x) = e_{+1}^{+1}(x) = 1$.

(*iv*) For $x \in D_\alpha^{\alpha+1}$, $e_{-1}^{-1}(x) = e_0^0(x) = e_{+1}^0(x) = e_{+1}^{+1}(x) = 1$.

(v) For $x \in D^{\alpha+1}_{\alpha+1}$, one of the followings holds.

(a) $e^{-1}_{-1}(x) = e^{0}_{0}(x) = e^{0}_{+1}(x) = e^{+1}_{0}(x) = 1$,

(b) $e^{-1}_{0}(x) = e^{0}_{-1}(x) = e^{0}_{+1}(x) = e^{+1}_{0}(x) = 1$.

(vi) For $x \in D^{\alpha+1}_{\alpha+2}$, $\quad e^{-1}_{-1}(x) = e^{+1}_{-1}(x) = 1,\ e^{0}_{-1}(x) = 2$.

(vii) For $x \in D^{\alpha+2}_{\alpha+1}$, $\quad e^{-1}_{-1}(x) = e^{-1}_{+1}(x) = 1,\ e^{-1}_{0}(x) = 2$.

Now let $\overline{G}$ be the triangle graph of G. We shall use the notations defined in section 5.2. Take a vertex w in D^1_1 and put $\overline{u} = \{u, v, w\}$. Then every vertex $\overline{x} = \{x_1, x_2, x_3\}$ in $\overline{G}$ satisfies one of the followings (by rearranging the order of x_1, x_2, x_3).

(i) $\overline{x} = \overline{u}$

(ii) $x_1 \in D^{r}_{r-1}$, $\quad x_2, x_3 \in D^{r+1}_{r}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = r \quad (1 \leq r \leq \alpha)$.

(iii) $x_1 \in D^{r-1}_{r}$, $\quad x_2, x_3 \in D^{r}_{r+1}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = r \quad (1 \leq r \leq \alpha)$.

(iv) $x_1 \in D^{r}_{r}$, $\quad x_2, x_3 \in D^{r+1}_{r+1}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = r \quad (1 \leq r \leq \alpha)$.

(v) $x_1 \in D^{\alpha+1}_{\alpha}$, $x_2 \in D^{\alpha+1}_{\alpha+1}$, $\quad x_3 \in D^{\alpha+2}_{\alpha+1}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = \alpha+1$.

(vi) $x_1 \in D^{\alpha}_{\alpha+1}$, $x_2 \in D^{\alpha+1}_{\alpha+1}$, $\quad x_3 \in D^{\alpha+1}_{\alpha+2}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = \alpha+1$.

(vii) $x_1 \in D^{\alpha+1}_{\alpha+1}$, $x_2 \in D^{\alpha+1}_{\alpha+2}$, $\quad x_3 \in D^{\alpha+2}_{\alpha+1}$, $\quad \overline{\partial}(\overline{u}, \overline{x}) = \alpha+1$.

We define $\overline{\Gamma}_r(\overline{u}) = \{\, \overline{x} \mid \overline{\partial}(\overline{u}, \overline{x}) = r \,\}$. For $\overline{x} \in \overline{\Gamma}_r(\overline{u})$, we define

$$a(\overline{x}, \overline{u}) = |\ \overline{\Gamma}_1(\overline{x}) \cap \overline{\Gamma}_r(\overline{u})\ |,$$

$$b(\overline{x}, \overline{u}) = |\ \overline{\Gamma}_1(\overline{x}) \cap \overline{\Gamma}_{r+1}(\overline{u})\ |,$$

$$c(\overline{x}, \overline{u}) = |\ \overline{\Gamma}_1(\overline{x}) \cap \overline{\Gamma}_{r-1}(\overline{u})\ |.$$

Then we get the followings.

(i) For $\overline{x} = \overline{u}$, $\quad c(\overline{x}, \overline{u}) = a(\overline{x}, \overline{u}) = 0$, $\quad b(\overline{x}, \overline{u}) = 3$.

(ii) For $\overline{x} \in \overline{\Gamma}_r(\overline{u})$ $\ (1 \leq r \leq \alpha)$, $\ c(\overline{x}, \overline{u}) = 1, a(\overline{x}, \overline{u}) = 0, b(\overline{x}, \overline{u}) = 2$.

(iii) For $\overline{x} \in \overline{\Gamma}_{\alpha+1}(\overline{u})$, $\quad c(\overline{x}, \overline{u}) = 1,\ a(\overline{x}, \overline{u}) = 2,\ b(\overline{x}, \overline{u}) = 0$.

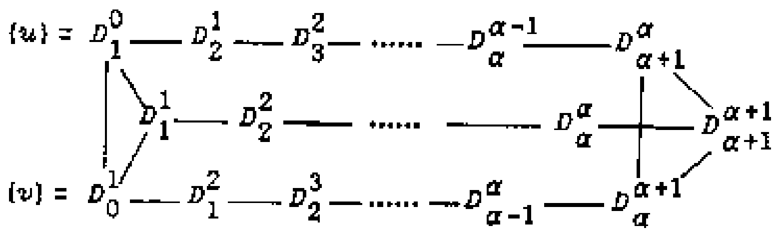
Remark that the values $a(\overline{x}, \overline{u}), b(\overline{x}, \overline{u}), c(\overline{x}, \overline{u})$ are depends only on r rather than the individual vertices $\overline{u}, \overline{x}$. Hence $\overline{G}$ is distance-regular with the intersection array

$$\left\{\begin{matrix} 0 & 1 & \cdots\cdots & 1 & 1 \\ 0 & 0 & \cdots\cdots & 0 & 2 \\ 3 & 2 & \cdots\cdots & 2 & 0 \end{matrix}\right\}$$

where the number of colums of type (1,0,2) is α. Hence G is a Moore graph.

Proposition 5.13. *If the intersection array of G is of type 2, the triangle graph $\overline{G}$ is a generalized polygon.*

Proof. We shall use the same notations in the proof of Proposition 5.12. In this case, the intersection diagram of G takes the following form.



The numbers are determined as follows by using lemmas in Chapter 3 and by using the fact that G is locally triangular.

(i) For $x \in D_1^1$, $e_0^{-1}(x) = e_{-1}^0(x) = 1$, $e_{+1}^{+1}(x) = 2$.

(ii) For $x \in D_{\tau+1}^{\tau}$, $D_\tau^{\tau+1}$, $(1 \leq \tau \leq \alpha-1)$ $e_{-1}^{-1}(x) = e_0^0(x) = 1$, $e_{+1}^{+1}(x) = 2$.

(iii) For $x \in D_{\alpha+1}^{\alpha}$, $e_{-1}^{-1}(x) = e_0^0(x) = e_{-1}^{+1}(x) = e_0^{+1}(x) = 1$.

(iv) For $x \in D_\alpha^{\alpha+1}$, $e_{-1}^{-1}(x) = e_0^0(x) = e_{+1}^{-1}(x) = e_{+1}^0(x) = 1$.

(v) For $x \in D_{\alpha+1}^{\alpha+1}$, $e_{-1}^{-1}(x) = e_0^0(x) = e_0^{-1}(x) = e_{-1}^0(x) = 1$.

We describe the proof of (v), others are easy. It is easy to show that one of the followings holds.

(a) $e_{-1}^{-1}(x) = e_0^0(x) = e_0^{-1}(x) = e_{-1}^0(x) = 1$,

(b) $e_0^{-1}(x) = e_{-1}^0(x) = 2$.

But we have $|D_\alpha^\alpha| = 2^{\alpha-1}$ and $|D_{\alpha+1}^{\alpha+1}| = 2^\alpha$. By counting the number of edges between D_α^α and $D_{\alpha+1}^{\alpha+1}$, we get that (b) does not occur, so (v) has been proved.

Let $\overline{u} = \{u, v, w\}$, where $D_1^1 = \{w\}$. Then every vertex $\overline{x}$ in $\overline{G}$ satisfies one of the followings.

(i) $\overline{x} = \overline{u}$

(ii) $x_1 \in D_{r-1}^r$, $x_2, x_3 \in D_r^{r+1}$, $\overline{\partial}(\overline{u}, \overline{x}) = r$ $(1 \leq r \leq \alpha)$.

(iii) $x_1 \in D_r^{r-1}$, $x_2, x_3 \in D_{r+1}^r$, $\overline{\partial}(\overline{u}, \overline{x}) = r$ $(1 \leq r \leq \alpha)$.

(iv) $x_1 \in D_r^r$, $x_2, x_3 \in D_{r+1}^{r+1}$, $\overline{\partial}(\overline{u}, \overline{x}) = r$ $(1 \leq r \leq \alpha)$.

(v) $x_1 \in D_{\alpha+1}^\alpha$, $x_2 \in D_\alpha^{\alpha+1}$, $x_3 \in D_{\alpha+1}^{\alpha+1}$, $\overline{\partial}(\overline{u}, \overline{x}) = \alpha+1$.

Then it is not difficult to show that $\overline{G}$ is a distance-regular graph with the following intersection array.

$$\left\{ \begin{matrix} 0 & 1 & \cdots\cdots & 1 & 3 \\ 0 & 0 & \cdots\cdots & 0 & 0 \\ 3 & 2 & \cdots\cdots & 2 & 0 \end{matrix} \right\}$$

Hence $\overline{G}$ is a generalized polygon.

Proof of Theorem 5.1. Since the girth of G is three, the intersection number a_1 cannot be zero, so $a_1 = 1$, 2 or 3. If $a_1 = 3$, it will be directly checked that G is isomorphic to the complete graph K_5. If $a_1 = 2$, it is not difficult to show that G is isomorphic to Octahedron. We leave the proofs of these facts to the reader.

So we may assume $a_1 = 1$. Then the triangle graph $\overline{G}$ is a Moore graph or a generarized polygon by Proposition 5.12 and Proposition 5.13. Then G is isomorphic to the line graph of $\overline{G}$ by Lemma 5.3.

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