Tridiagonal pairs and algebraic graph theory

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Contents

Let X denote a nonempty finite set.

$\text{Mat}_X(\mathbb{C})$ denotes the $\mathbb{C}$-algebra consisting of the matrices over $\mathbb{C}$ that have rows and columns indexed by X.

$V = \mathbb{C}X$ denotes the vector space over $\mathbb{C}$ consisting of column vectors with rows indexed by X.

$\text{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

Preliminaries, cont.

Endow V with a Hermitean inner product

$$\langle u, v \rangle = u^t \overline{v} \qquad\qquad (u, v \in V)$$

For each $x \in X$ let $\hat{x}$ denote the vector in V that has a 1 in coordinate x and 0 in all other coordinates.

Observe $\{\hat{x} | x \in X\}$ is an orthonormal basis for V.

The graph Γ

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, and path-length distance function ∂.

For an integer $i \geq 0$ and $x \in X$ let

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

The graphs of interest

Our main case of interest is when Γ is “highly regular” in a certain way.

A good example to keep in mind is the D-**dimensional hypercube**, also called the **binary Hamming graph** $H(D, 2)$.

Note that $H(2, 2)$ is a 4-cycle; this will be used as a running example.

The adjacency matrix

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the $(0,1)$-adjacency matrix of Γ.

For $x \in X$,

$$A\hat{x} = \sum_{y \in \Gamma(x)} \hat{y}$$

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Example

For $H(2,2)$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The adjacency algebra M

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.

M is called the **adjacency algebra** of Γ.

M is commutative and semisimple.

Example

For $H(2,2)$, M has a basis

$$I, A, J$$

where the matrix J has every entry 1.

The primitive idempotents of Γ

Since M is semisimple it has basis $\{E_i\}_{i=0}^d$ such that

$$
\begin{aligned}
E_i E_j &= \delta_{ij} E_i \qquad\qquad (0 \le i, j \le d), \\
I &= \sum_{i=0}^{d} E_i.
\end{aligned}
$$

We call $\{E_i\}_{i=0}^d$ the **primitive idempotents** of Γ.

The primitive idempotents of Γ

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$$\begin{aligned} E_i E_j &= \delta_{ij} E_i \qquad\qquad (0 \leq i, j \leq d), \\ I &= \sum_{i=0}^{d} E_i. \end{aligned}$$

We call $\{E_i\}_{i=0}^d$ the **primitive idempotents** of Γ.
Write

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

For $0 \leq i \leq d$ the scalar θ_i is the **eigenvalue** of A for E_i.

The primitive idempotents, cont.

Example

For $H(2,2)$ we have $\theta_0 = 2$, $\theta_1 = 0$, $\theta_2 = -2$. Moreover

$$E_0 = 1/4J,$$

$$E_1 = 1/4 \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix},$$

$$E_2 = 1/4 \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The eigenspaces of A

The vector space V decomposes as

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum)}$$

For $0 \leq i \leq d$ the space $E_i V$ is the **eigenspace of** A associated with the eigenvalue θ_i.

the matrix E_i represents the **orthogonal projection** onto $E_i V$.

Until further notice

$$\textbf{fix } x \in X.$$

We call x the **base vertex**.

Define $D = D(x)$ by

$$D = \max\{\partial(x, y) \mid y \in X\}$$

We call D the **diameter of Γ with respect to** x.

The dual primitive idempotents of Γ

Until further notice

$$\textbf{fix } x \in X.$$

We call x the **base vertex**.

Define $D = D(x)$ by

$$D = \max\{\partial(x,y) \mid y \in X\}$$

We call D the **diameter of Γ with respect to** x.

For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y,y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i. \end{cases} \qquad (y \in X).$$

The dual primitive idempotents, cont.

Example

For $H(2,2)$,

$$E_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_1^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $\{E_i^*\}_{i=0}^D$ satisfy

$$\begin{aligned} E_i^* E_j^* &= \delta_{ij} E_i^* \qquad\qquad (0 \le i, j \le D), \\ I &= \sum_{i=0}^D E_i^*. \end{aligned}$$

We call $\{E_i^*\}_{i=0}^D$ the **dual primitive idempotents of Γ with respect to** x.

The dual adjacency algebra

The $\{E_i^*\}_{i=0}^D$ form a basis for a semisimple commutative subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ denoted $M^* = M^*(x)$.

We call M^* the **dual adjacency algebra of Γ with respect to x**.

The subconstituents of Γ

The vector space V decomposes as

$$V = \sum_{i=0}^{D} E_i^* V \qquad \text{(orthogonal direct sum)}$$

The above summands are the common eigenspaces for M^*.

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The vector space V decomposes as

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The above summands are the common eigenspaces for M^*.

These eigenspaces have the following combinatorial interpretation. For $0 \leq i \leq D$,

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in \Gamma_i(x)\}$$

We call $E_i^* V$ the **ith subconstituent of Γ with respect to** x.

The matrix E_i^* represents the **orthogonal projection** onto $E_i^* V$.

So far we defined the adjacency algebra M and the dual adjacency algebra M^*. We now combine M and M^* to get a larger algebra.

Definition

(Ter 92) Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. T is called the **subconstituent algebra of Γ with respect to x**.

T is finite-dimensional.

T is noncommutative in general.

T is semisimple

T is semi-simple because it is closed under the conjugate-transpose map.

So by the Wedderburn theory the algebra T is isomorphic to a direct sum of matrix algebras.

Example

For $H(2,2)$,

$$T \simeq \mathrm{Mat}_3(\mathbb{C}) \oplus \mathbb{C}.$$

Moreover $\dim(T) = 10$.

T is semisimple, cont.

Example

(Junie Go 2002) For $H(D, 2)$,

$$T \simeq \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-1}(\mathbb{C}) \oplus \mathrm{Mat}_{D-3}(\mathbb{C}) \oplus \cdots$$

Moreover $\dim(T) = (D+1)(D+2)(D+3)/6$.

T-modules

We mentioned that T is closed under the conjugate-transpose map.

So for each T-module $W \subseteq V$, its orthogonal complement $W^\perp$ is also a T-module.

Therefore V decomposes into an orthogonal direct sum of irreducible T-modules.

Problem

(i) How does the above decomposition reflect the combinatorial properties of Γ? (ii) For which graphs Γ is the above decomposition particularly nice?

We now describe a family of graphs for which the irreducible T-modules are nice.

These graphs possess a certain matrix called a **dual adjacency matrix**.

To motivate this concept we consider some relations in T.

Some relations in T

By the triangle inequality

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

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where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

This is reformulated as follows.

Lemma

For $0 \leq i, j \leq D$,

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j] > 1.$$

The dual adjacency matrix

Definition

Referring to the graph Γ, consider a matrix $A^* \in \mathrm{Mat}_X(\mathbb{C})$ that satisfies both conditions below:

(i) A^* generates M^*;

(ii) For $0 \leq i, j \leq d$,

$$E_i A^* E_j = 0 \quad \text{if } |i - j] > 1.$$

We call A^* a **dual adjacency matrix** (with respect to x and the given ordering $\{E_i\}_{i=0}^{d}$ of the primitive idempotents).

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A dual adjacency matrix A^* is **diagonal**.

The dual adjacency matrix

A dual adjacency matrix A^* acts on the eigenspaces of A as follows.

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

Example

$H(2,2)$ has a dual adjacency matrix

$$A^* = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

An example

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Example

$H(D,2)$ has a dual adjacency matrix

$$A^* = \sum_{i=0}^{D} (D-2i) E_i^*.$$

More examples

The following graphs have a dual adjacency matrix.

Any **strongly-regular** graph.

Any Q-**polynomial distance-regular** graph, for instance:

- cycle
- Hamming graph
- Johnson graph
- Grassman graph
- Dual polar spaces
- Bilinear forms graph
- Alternating forms graph
- Hermitean forms graph
- Quadratic forms graph

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See the book **Distance-Regular Graphs** by Brouwer, Cohen, Neumaier.

How A and A^* are related

To summarize so far, for our graph Γ the adjacency matrix A and any dual adjacency matrix A^* generate T. Moreover they act on each other's eigenspaces in the following way:

$$AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \qquad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

$$A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \leq i \leq d),$$

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$$A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$.

To clarify this situation we reformulate it as a problem in linear algebra.

We now define a linear-algebraic object called a **TD pair**.

From now on $\mathbb{F}$ denotes a field.

V will denote a vector space over $\mathbb{F}$ with finite positive dimension.

We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$.

Definition of a Tridiagonal pair

We say the pair A, A^* is a **TD pair** on V whenever (1)–(4) hold below.

1. Each of A, A^* is diagonalizable on V.
2. There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

   where $V_{-1} = 0$, $V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^D$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq D),$$

   where $V_{-1}^* = 0$, $V_{D+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^* W \subseteq W$ and $W \neq 0$ and $W \neq V$.

Referring to our definition of a TD pair,

it turns out $d = D$; we call this common value the **diameter** of the pair.

Each irreducible T-module gives a TD pair

Briefly returning to the graph Γ, the adjacency matrix and any dual adjacency matrix act on each irreducible T-module as a TD pair.

This motivates us to understand TD pairs.

Special case: Leonard pairs

In our study of TD pairs we begin with a special case called a **Leonard pair**.

A Leonard pair is a TD pair for which the eigenspaces V_i and V_i^* all have dimension 1.

Special case: Leonard pairs

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A Leonard pair is a TD pair for which the eigenspaces V_i and V_i^* all have dimension 1.

The Leonard pairs are classified up to isomorphism (Ter 2001).

The solutions correspond to a family of orthogonal polynomials that make up the **terminating branch of the Askey scheme**.

This family consists of the q-**Racah polynomials** and their relatives.

The classification of TD pairs

We now turn to general TD pairs.

After 10 years of work and several dozen papers, my collaborators Tatsuro Ito, Kazumasa Nomura and I have classified up to isomorphism the TD pairs over an algebraically closed field.

To be precise, we classified up to isomorphism a more general family of TD pairs said to be **sharp**.

T. Ito, K. Nomura, P. Terwilliger
The classification of the sharp tridiagonal pairs.
Linear Algebra Appl. Submitted.

I will describe this result shortly.

TD pairs and TD systems

When working with a TD pair, it is helpful to consider a closely related object called a **TD system**.

We will define a TD system over the next few slides.

Referring to our definition of a TD pair,

An ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

Standard orderings

Referring to our definition of a TD pair,

An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

In this case, the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard.

A similar discussion applies to A^*.

Primitive idempotents

Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** E is the projection onto that eigenspace.

In other words $E - I$ vanishes on the eigenspace and E vanishes on all the other eigenspaces.

TD systems

Definition

By a **TD system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

that satisfies the following:

1. A, A^* is a TD pair on V.
2. $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A.
3. $\{E_i^*\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A^*.

Until further notice we fix a TD system Φ as above.

The eigenvalues

For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with the eigenspace E_iV (resp. E^*_iV).

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

A three-term recurrence

Theorem (Ito+Tanabe+T, 2001)

The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \le i \le d-1$.

Let $\beta + 1$ denote the common value of the above expressions.

Solving the recurrence

For the above recurrence the "simplest" solution is

$$\theta_i = d - 2i \ \ (0 \le i \le d),$$
$$\theta^*_i = d - 2i \ (0 \le i \le d).$$

In this case $\beta = 2$.

For this solution our TD system Φ is said to have **Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence another solution is

$$\theta_i = q^{d-2i} \ \ (0 \leq i \leq d),$$
$$\theta^*_i = q^{d-2i} \ (0 \leq i \leq d),$$
$$q \neq 0, \ \ q^2 \neq 1, \ \ q^2 \neq -1.$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence the "most general" solution is

$$\theta_i = a + bq^{2i-d} + cq^{d-2i} \quad (0 \le i \le d),$$
$$\theta^*_i = a^* + b^*q^{2i-d} + c^*q^{d-2i} \; (0 \le i \le d),$$
$$q,\ a,\ b,\ c,\ a^*,\ b^*,\ c^* \in \overline{\mathbb{F}},$$
$$q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1, \quad bb^*cc^* \neq 0.$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Racah type**.

Some notation

For later use we define some polynomials in an indeterminate λ.

For $0 \leq i \leq d$,

$$
\begin{aligned}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).
\end{aligned}
$$

Note that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

The shape

It is known that for $0 \leq i \leq d$ the eigenspaces E_iV, E^*_iV have the same dimension; we denote this common dimension by ρ_i.

Lemma (Ito+Tanabe+T, 2001)

The sequence $\{\rho_i\}_{i=0}^d$ *is* **symmetric** *and* **unimodal***; that is*

$$\rho_i = \rho_{d-i} \qquad (0 \leq i \leq d),$$
$$\rho_{i-1} \leq \rho_i \qquad (1 \leq i \leq d/2).$$

We call the sequence $\{\rho_i\}_{i=0}^d$ the **shape** of Φ.

A bound on the shape

Theorem (Ito+Nomura+T, 2009)

The shape $\{\rho_i\}_{i=0}^d$ *of* Φ *satisfies*

$$\rho_i \leq \rho_0 \binom{d}{i} \qquad (0 \leq i \leq d).$$

Moreover if $\mathbb{F}$ *is algebraically closed then* $\rho_0 = 1$.

The tridiagonal relations

Theorem (Ito+Tanabe+T, 2001)

For our TD system Φ *there exist scalars* $\gamma, \gamma^*, \varrho, \varrho^*$ *in* $\mathbb{F}$ *such that*

$$\begin{aligned} A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ = \quad \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ = \quad \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

The above equations are called the **tridiagonal relations**.

In the Krawtchouk case the tridiagonal relations become the **Dolan-Grady relations**

$$[A,[A,[A,A^*]]] = 4[A,A^*],$$

$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A].$$

Here $[r,s] = rs - sr$.

The q-Serre relations

In the q-Krawtchouk case the tridiagonal relations become the cubic q-**Serre relations**

$$
\begin{aligned}
A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 &= 0,\\
A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} &= 0.
\end{aligned}
$$

$$
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots
$$

The sharp case

At this point it is convenient to make an assumption about our TD system Φ.

Φ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^{d}$ is the shape of Φ.

The sharp case

At this point it is convenient to make an assumption about our TD system Φ.

Φ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^{d}$ is the shape of Φ.

If the ground field $\mathbb{F}$ is algebraically closed then Φ is sharp.

Until further notice assume Φ is sharp.

The split decomposition

For $0 \leq i \leq d$ define

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V).$$

It is known that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

and for $0 \leq i \leq d$ both

$$\begin{aligned} U_0 + \cdots + U_i &= E_0^* V + \cdots + E_i^* V, \\ U_i + \cdots + U_d &= E_i V + \cdots + E_d V. \end{aligned}$$

We call the sequence $\{U_i\}_{i=0}^d$ the **split decomposition** of V with respect to Φ.

The split decomposition, cont.

Theorem (Ito+Tanabe+T, 2001)

For $0 \leq i \leq d$ *both*

$$\begin{aligned}(A - \theta_i I)U_i &\subseteq U_{i+1},\\ (A^* - \theta^*_i I)U_i &\subseteq U_{i-1},\end{aligned}$$

where $U_{-1} = 0$, $U_{d+1} = 0$.

Observe that for $0 \leq i \leq d$,

$$\begin{aligned}(A-\theta_{i-1}I)\cdots(A-\theta_1 I)(A-\theta_0 I)U_0 &\subseteq U_i,\\ (A^*-\theta^*_1 I)\cdots(A^*-\theta^*_{i-1}I)(A^*-\theta^*_i I)U_i &\subseteq U_0.\end{aligned}$$

Therefore U_0 is invariant under

$$(A^*-\theta^*_1 I)\cdots(A^*-\theta^*_i I)(A-\theta_{i-1}I)\cdots(A-\theta_0 I).$$

Let ζ_i denote the corresponding eigenvalue and note that $\zeta_0 = 1$.

We call the sequence $\{\zeta_i\}_{i=0}^d$ the **split sequence** of Φ.

The split sequence $\{\zeta_i\}_{i=0}^d$ is characterized as follows.

Lemma (Nomura+T, 2007)

For $0 \le i \le d$,

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

A restriction on the split sequence

The split sequence $\{\zeta_i\}_{i=0}^d$ satisfies two inequalities.

Lemma (Ito+Tanabe+T, 2001)

$$
\begin{aligned}
0 &\neq E_0^* E_d E_0^*, \\
0 &\neq E_0^* E_0 E_0^*.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
0 &\neq \zeta_d, \\
0 &\neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.
\end{aligned}
$$

Lemma (Ito+ Nomura+T, 2008)

The TD system Φ is determined up to isomorphism by the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d).$$

We call this sequence the **parameter array** of Φ.

Lemma (Ito+ Nomura+T, 2008)

The TD system Φ is determined up to isomorphism by the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d).$$

We call this sequence the **parameter array** of Φ.

We are now ready to state our classification theorem.

A classification of sharp tridiagonal systems

Theorem (Ito+Nomura+T, 2009)

Let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ (1) *denote a sequence of scalars in* $\mathbb{F}$. *Then there exists a sharp TD system* Φ *over* $\mathbb{F}$ *with parameter array (1) if and only if:*

(i) $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta^*_j$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) *the expressions* $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$, $\frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$ *are equal and independent of* i *for* $2 \leq i \leq d-1$;

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Suppose (i)–(iii) *hold. Then* Φ *is unique up to isomorphism of TD systems.*

The classification: proof summary

In the proof the hard part is to construct a TD system with given parameter array of q-Racah type.

To do this we make use of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$.

Using the parameter array we identify two elements in $U_q(\widehat{\mathfrak{sl}_2})$ that satisfy some tridiagonal relations.

We let these elements act on a certain $U_q(\widehat{\mathfrak{sl}_2})$-module of the form $W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where each W_i is an evaluation module of dimension 2.

This action yields the desired TD system after a reduction process.

Summary

In Part I we discussed the **subconstituent algebra** T of a graph Γ and considered how the standard module V decomposes into a direct sum of irreducible T-modules.

We identified a class of graphs for which the irreducible T-modules are nice; these graphs possess a **dual adjacency matrix**.

For these graphs the adjacency matrix and dual adjacency matrix act on each irreducible T-module as a **TD pair**.

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In Part I we discussed the **subconstituent algebra** T of a graph Γ and considered how the standard module V decomposes into a direct sum of irreducible T-modules.

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For these graphs the adjacency matrix and dual adjacency matrix act on each irreducible T-module as a **TD pair**.

In Part II we considered general TD pairs. We discussed the **eigenvalues, dual eigenvalues, shape, tridiagonal relations**, and **parameter array**.

We then classified up to isomorphism the TD pairs over an algebraically closed field. In the future we hope to apply this classification to the study of graphs.

Thank you for your attention!

THE END