A classification of sharp tridiagonal pairs

Tatsuro Ito Kazumasa Nomura Paul Terwilliger

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We will describe its features such as the eigenvalues, dual eigenvalues, shape, tridiagonal relations, split decomposition and parameter array.

We will then define an algebra $\mathbb{T}$ by generators and relations, and prove a theorem about its structure called the μ-**Theorem**.

We will use the μ-Theorem to obtain a **Classification Theorem** for sharp tridiagonal pairs.

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$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

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Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

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The Definition of a Leonard Pair

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Definition

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a **Leonard pair** on V, we mean a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy both conditions below.

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
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In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (1), (2) above.

Example of a Leonard pair

For any integer $d \geq 0$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix},$$

$$A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than d.

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is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than d.

Reason: There exists an invertible matrix P such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q-Racah,
q-Hahn,
dual q-Hahn,
q-Krawtchouk,
dual q-Krawtchouk,
quantum q-Krawtchouk,
affine q-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans ($\mathrm{char}(\mathbb{F}) = 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

Recommended reading

The theory of Leonard pairs is summarized in

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; `arXiv:math.QA/0408390`.

Tridiagonal pairs

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As before, we consider a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$.

Definition of a Tridiagonal pair

We say the pair A, A^* is a **TD pair** on V whenever (1)–(4) hold below.

1. Each of A, A^* is diagonalizable on V.
2. There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that
$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$
where $V_{-1} = 0,\ V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that
$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$
where $V_{-1}^* = 0,\ V_{\delta+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^* W \subseteq W$ and $W \neq 0$ and $W \neq V$.

Referring to our definition of a TD pair,

it turns out $d = \delta$; we call this common value the **diameter** of the pair.

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A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces V_i and V_i^* all have dimension 1.

Origins

The concept of a TD pair originated in **algebraic graph theory**, or more precisely, the theory of Q-**polynomial distance-regular graphs**. See

T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192;
`arXiv:math.CO/0406556`.

When working with a TD pair, it is helpful to consider a closely related object called a **TD system**.

We will define a TD system over the next few slides.

Standard orderings

Referring to our definition of a TD pair,

An ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

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In this case, the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard.

A similar discussion applies to A^*.

Primitive idempotents

Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** E is the projection onto that eigenspace.

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Given an eigenspace of a diagonalizable linear transformation, the corresponding **primitive idempotent** E is the projection onto that eigenspace.

In other words $E - I$ vanishes on the eigenspace and E vanishes on all the other eigenspaces.

Definition

By a **TD system** on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies the following:

1. A, A^* is a TD pair on V.
2. $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.
3. $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

Until further notice we fix a TD system Φ as above.

The eigenvalues

For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with the eigenspace E_iV (resp. E^*_iV).

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For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with the eigenspace $E_i V$ (resp. $E^*_i V$).

We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

A three-term recurrence

Theorem (Ito+Tanabe+T, 2001)

The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

Let $\beta + 1$ denote the common value of the above expressions.

Solving the recurrence

For the above recurrence the "simplest" solution is

$$\theta_i = d - 2i \;\; (0 \le i \le d),$$
$$\theta^*_i = d - 2i \; (0 \le i \le d).$$

In this case $\beta = 2$.

For this solution our TD system Φ is said to have **Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence another solution is

$$\theta_i = q^{d-2i} \quad (0 \leq i \leq d),$$
$$\theta^*_i = q^{d-2i} \; (0 \leq i \leq d),$$
$$q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1.$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Krawtchouk type**.

Solving the recurrence, cont.

For the above recurrence the "most general" solution is

$$
\begin{aligned}
&\theta_i = a + bq^{2i-d} + cq^{d-2i} \ \ (0 \leq i \leq d),\\
&\theta^*_i = a^* + b^*q^{2i-d} + c^*q^{d-2i} \ (0 \leq i \leq d),\\
&q,\ a,\ b,\ c,\ a^*,\ b^*,\ c^* \in \overline{\mathbb{F}},\\
&q \neq 0,\ \ q^2 \neq 1,\ \ q^2 \neq -1,\ \ bb^*cc^* \neq 0.
\end{aligned}
$$

In this case $\beta = q^2 + q^{-2}$.

For this solution Φ is said to have q-**Racah type**.

Some notation

For later use we define some polynomials in an indeterminate λ.

For $0 \leq i \leq d$,

$$\begin{aligned}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).
\end{aligned}$$

Note that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

The shape

It is known that for $0 \le i \le d$ the eigenspaces $E_i V$, $E_i^* V$ have the same dimension; we denote this common dimension by ρ_i.

Lemma (Ito+Tanabe+T, 2001)

The sequence $\{\rho_i\}_{i=0}^d$ *is* **symmetric** *and* **unimodal**; *that is*

$$\rho_i = \rho_{d-i} \qquad (0 \le i \le d),$$
$$\rho_{i-1} \le \rho_i \qquad (1 \le i \le d/2).$$

We call the sequence $\{\rho_i\}_{i=0}^d$ the **shape** of Φ.

A bound on the shape

Theorem (Ito+Nomura+T, 2009)

The shape $\{\rho_i\}_{i=0}^d$ *of* Φ *satisfies*

$$\rho_i \leq \rho_0 \binom{d}{i} \qquad (0 \leq i \leq d).$$

The parameter ρ_0

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We will explain this after a few slides.

Some relations

Lemma

Our TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *satisfies the following relations:*

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^* \qquad 0 \le i, j \le d,$$

$$1 = \sum_{i=0}^d E_i, \qquad 1 = \sum_{i=0}^d E_i^*,$$

$$A = \sum_{i=0}^d \theta_i E_i, \qquad A^* = \sum_{i=0}^d \theta_i^* E_i^*,$$

$$E_i^* A^k E_j^* = 0 \quad \text{if} \;\; k < |i - j| \qquad 0 \le i, j, k \le d,$$

$$E_i {A^*}^k E_j = 0 \quad \text{if} \;\; k < |i - j| \qquad 0 \le i, j, k \le d.$$

We call these last two equations the **triple product relations**.

Given the relations on the previous slide, it is natural to consider the algebra generated by A; A^*; $\{E_i\}_{i=0}^d$; $\{E_i^*\}_{i=0}^d$.

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Consider the space $E_0^* T E_0^*$.

Observe that $E_0^* T E_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity E_0^*.

The algebra $E_0^* T E_0^*$

Theorem (Ito+Nomura+T, 2007)

(i) The $\mathbb{F}$-algebra $E_0^* T E_0^*$ is commutative and generated by

$$E_0^* A^i E_0^* \qquad 1 \leq i \leq d.$$

(ii) $E_0^* T E_0^*$ has no zero-divisors; in other words it is a field.
(iii) Viewing this field as a field extension of $\mathbb{F}$, the index is ρ_0.

The parameter ρ_0

Corollary (Ito+Nomura+T, 2007)

If $\mathbb{F}$ is algebraically closed then $\rho_0 = 1$.

We now consider some more relations in T.

Theorem (Ito+Tanabe+T, 2001)

For our TD system Φ *there exist scalars* $\gamma, \gamma^*, \varrho, \varrho^*$ *in* $\mathbb{F}$ *such that*

$$\begin{aligned} & A^3A^* - (\beta+1)A^2A^*A + (\beta+1)AA^*A^2 - A^*A^3 \\ = \quad & \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A), \end{aligned}$$

$$\begin{aligned} & A^{*3}A - (\beta+1)A^{*2}AA^* + (\beta+1)A^*AA^{*2} - AA^{*3} \\ = \quad & \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*). \end{aligned}$$

The above equations are called the **tridiagonal relations**.

In the Krawtchouk case the tridiagonal relations become the **Dolan-Grady relations**

$$[A,[A,[A,A^*]]] = 4[A,A^*],$$

$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A].$$

Here $[r,s] = rs - sr$.

The q-Serre relations

In the q-Krawtchouk case the tridiagonal relations become the cubic q-**Serre relations**

$$\begin{aligned} A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 &= 0, \\ A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} &= 0. \end{aligned}$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots$$

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Φ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^{d}$ is the shape of Φ.

If the ground field $\mathbb{F}$ is algebraically closed then Φ is sharp.

Until further notice assume Φ is sharp.

The split decomposition

For $0 \leq i \leq d$ define

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V).$$

It is known that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

and for $0 \leq i \leq d$ both

$$\begin{aligned} U_0 + \cdots + U_i &= E_0^* V + \cdots + E_i^* V, \\ U_i + \cdots + U_d &= E_i V + \cdots + E_d V. \end{aligned}$$

We call the sequence $\{U_i\}_{i=0}^d$ the **split decomposition** of V with respect to Φ.

The split decomposition, cont.

Theorem (Ito+Tanabe+T, 2001)

For $0 \leq i \leq d$ *both*

$$\begin{aligned} (A - \theta_i I)U_i &\subseteq U_{i+1}, \\ (A^* - \theta_i^* I)U_i &\subseteq U_{i-1}, \end{aligned}$$

where $U_{-1} = 0$, $U_{d+1} = 0$.

The split sequence, cont.

Observe that for $0 \le i \le d$,

$$\begin{aligned}(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I) U_0 &\subseteq U_i, \\ (A^* - \theta^*_1 I) \cdots (A^* - \theta^*_{i-1} I)(A^* - \theta^*_i I) U_i &\subseteq U_0.\end{aligned}$$

Therefore U_0 is invariant under

$$(A^* - \theta^*_1 I) \cdots (A^* - \theta^*_i I)(A - \theta_{i-1} I) \cdots (A - \theta_0 I).$$

Let ζ_i denote the corresponding eigenvalue and note that $\zeta_0 = 1$.

We call the sequence $\{\zeta_i\}_{i=0}^{d}$ the **split sequence** of Φ.

Characterizing the split sequence

The split sequence $\{\zeta_i\}_{i=0}^d$ is characterized as follows.

Lemma (Nomura+T, 2007)

For $0 \leq i \leq d$,

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

A restriction on the split sequence

The split sequence $\{\zeta_i\}_{i=0}^d$ satisfies two inequalities.

Lemma (Ito+Tanabe+T, 2001)

$$
\begin{aligned}
0 &\neq E_0^* E_d E_0^*, \\
0 &\neq E_0^* E_0 E_0^*.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
0 &\neq \zeta_d, \\
0 &\neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.
\end{aligned}
$$

The parameter array

Lemma (Ito+ Nomura+T, 2008)

The TD system Φ is determined up to isomorphism by the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d).$$

We call this sequence the **parameter array** of Φ.

The μ-Theorem

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We will define $\mathbb{T}$ shortly.

Feasible sequences

Definition

Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$. This sequence is called **feasible** whenever both

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) the expressions $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$, $\frac{\theta_{i-2}^*-\theta_{i+1}^*}{\theta_{i-1}^*-\theta_i^*}$ are equal and independent of i for $2 \leq i \leq d-1$.

Definition

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$. Let $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ denote the $\mathbb{F}$-algebra defined by generators a, $\{e_i\}_{i=0}^d$, a^*, $\{e^*_i\}_{i=0}^d$ and relations

$$e_i e_j = \delta_{i,j} e_i, \quad e^*_i e^*_j = \delta_{i,j} e^*_i \qquad 0 \le i, j \le d,$$

$$1 = \sum_{i=0}^d e_i, \qquad 1 = \sum_{i=0}^d e^*_i,$$

$$a = \sum_{i=0}^d \theta_i e_i, \qquad a^* = \sum_{i=0}^d \theta^*_i e^*_i,$$

$$e^*_i a^k e^*_j = 0 \quad \text{if} \;\; k < |i - j| \qquad 0 \le i, j, k \le d,$$

$$e_i {a^*}^k e_j = 0 \quad \text{if} \;\; k < |i - j| \qquad 0 \le i, j, k \le d.$$

Over the next few slides, we explain how TD systems are related to finite-dimensional irreducible $\mathbb{T}$-modules.

Lemma

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ where $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Then there exists a unique $\mathbb{T}$-module structure on V such that a, e_i, a^*, e_i^* acts as A, E_i, A^*, E_i^* respectively. This $\mathbb{T}$-module is irreducible.

Lemma

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ *and write* $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$. *Let* V *denote a finite-dimensional irreducible* $\mathbb{T}$*-module.*

(i) *There exist nonnegative integers* r, δ $(r + \delta \le d)$ *such that for* $0 \le i \le d$,

$$e_i^* V \neq 0 \quad \text{if and only if} \quad r \le i \le r + \delta.$$

(ii) *There exist nonnegative integers* t, δ^* $(t + \delta^* \le d)$ *such that for* $0 \le i \le d$,

$$e_i V \neq 0 \quad \text{if and only if} \quad t \le i \le t + \delta^*.$$

(iii) $\delta = \delta^*$.

(iv) *The sequence* $(a; \{e_i\}_{i=t}^{t+\delta}; a^*; \{e_i^*\}_{i=r}^{r+\delta})$ *acts on* V *as a TD system of diameter* δ.

The structure of $\mathbb{T}$

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ and consider the $\mathbb{F}$-algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$.

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Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})$ and consider the $\mathbb{F}$-algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$.

As we did with T we consider the space $e_0^* \mathbb{T} e_0^*$.

Observe that $e_0^* \mathbb{T} e_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity e_0^*.

Let $\{\lambda_i\}_{i=1}^d$ denote mutually commuting indeterminates.

Let $\mathbb{F}[\lambda_1, \ldots, \lambda_d]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\{\lambda_i\}_{i=1}^d$ that have all coefficients in $\mathbb{F}$.

The μ-Theorem

Theorem (Ito+Nomura+T, 2009)

There exists an $\mathbb{F}$-algebra isomorphism

$$\mathbb{F}[\lambda_1, \ldots, \lambda_d] \rightarrow e_0^* \mathbb{T} e_0^*$$

that sends

$$\lambda_i \mapsto e_0^* a^i e_0^*$$

for $1 \leq i \leq d$.

Proof summary: We first verify the result assuming p has q-Racah type. To do this we make use of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. We identify two elements in $U_q(\hat{\mathfrak{sl}}_2)$ that satisfy the tridiagonal relations. We let these elements act on $U_q(\hat{\mathfrak{sl}}_2)$-modules of the form $W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where each W_i is an evaluation module of dimension 2. Each of these actions gives a TD system of q-Racah type which in turn yields a $\mathbb{T}$-module. The resulting supply of $\mathbb{T}$-modules is sufficiently rich to contradict the existence of an algebraic relation among $\{e_0^* a^i e_0^*\}_{i=1}^d$.

We then remove the assumption that p has q-Racah type. In this step the main ingredient is to show that for any polynomial h over $\mathbb{F}$ in $2d+2$ variables, if $h(p) = 0$ under the assumption that p is q-Racah, then $h(p) = 0$ without the assumption.

A classification of sharp tridiagonal systems

Theorem (Ito+Nomura+T, 2009)

Let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ (1) *denote a sequence of scalars in* $\mathbb{F}$. *Then there exists a sharp TD system* Φ *over* $\mathbb{F}$ *with parameter array (1) if and only if:*

(i) $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta^*_j$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) *the expressions* $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$, $\frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$ *are equal and independent of* i *for* $2 \leq i \leq d-1$;

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Suppose (i)–(iii) *hold. Then* Φ *is unique up to isomorphism of TD systems.*

The classification: proof summary

Proof ("only if"): By our previous remarks.
Proof ("if"): Consider the algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ where $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$.

By the μ-Theorem $e_0^* \mathbb{T} e_0^*$ is a polynomial algebra.

Therefore $e_0^* \mathbb{T} e_0^*$ has a 1-dimensional module on which

$$e_0^* \tau_i(a) e_0^* = \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

for $1 \leq i \leq d$.

The above 1-dimensional $e_0^* \mathbb{T} e_0^*$-module induces a $\mathbb{T}$-module V which turns out to be finite-dimensional; by construction $e_0^* V$ has dimension 1.

One checks that the $\mathbb{T}$-module V has a unique maximal proper submodule M.

The classification: proof summary, cont.

Consider the irreducible $\mathbb{T}$-module V/M.

By the inequalities in (iii),

$$e_0^* e_d e_0^* \neq 0, \qquad e_0^* e_0 e_0^* \neq 0$$

on V/M.

Therefore each of e_0, e_d is nonzero on V/M.

Now the $\mathbb{T}$-generators $(a; \{e_i\}_{i=0}^d; a^*; \{e_i^*\}_{i=0}^d)$ act on V/M as a sharp TD system of diameter d.

One checks that this TD system has the desired parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.

QED

Summary

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Thank you for your attention!

THE END